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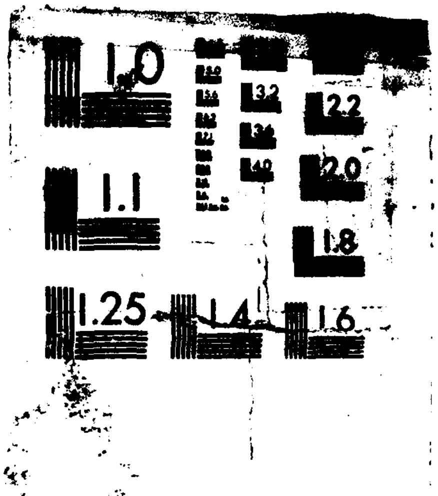

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Progress is described on a project whose goal is the development of practical tools for evaluating the reliability and availability of fault-tolerant control or sensor systems. The approach relies on the generation of a Markovian model for the behavior of the system. The project entails the investigation of approximate techniques for deriving results from these models. The basic idea is that the time-behavior of the model decomposes into two time scales where the results of interest occur in time frames intermediate to the two time scales. By modifying previous theory, an approximate evaluation scheme is developed and shown to be valid when the model possesses a decomposability property such that the decomposed classes obey a relatively weak sufficient condition. Examples are discussed. Also, means for generating the exact answers to which the results of the approximation will be compared are discussed.

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Approximate Evaluation of Reliability and Availability Via Perturbation Analysis

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I. INTRODUCTION

1.1 Motivation and Discussion of Problem

Reliability and availability have become two of the prime considerations in the design of control systems for a diverse group of applications that includes flight control systems for both aircraft and spacecraft. Considerable effort is now being devoted to the design of highly reliable control system components and to the design of fault-tolerant processors for online control computations. Despite the success of some of these efforts, the extremely high reliability goals that are becoming commonplace in the Air Force and elsewhere can often be met only by designing control systems with built-in component redundancy. The combination of a redundant system architecture and a redundancy management (RM) algorithm constitutes a fault-tolerant system design.

Predicting the performance of these designs is an important and difficult problem. The performance is judged by such quantities as the reliability, the availability, or some other probabilistic quantity such as average measurement accuracy or average regulation error. Calculating these quantities is an important problem because they represent the criteria by which various fault-tolerant system designs are judged. Such calculations are difficult because fault-tolerant systems are subject to random events, such as failures and RM decisions, that change the nature of operation of the system and therefore affect the values of the performance quantities.

Several papers and theses have introduced the concept of modelling the random behavior of a fault-tolerant system by generalized finite-state Markov models [1-6]. The states in these models characterize the status of the system in terms of the number of components that are operating, the number of these that are failed, and the status of the RM decisions. The transition behavior among these states must then be derived from the probabilistic behavior of component failures and of the RM decisions (including errors such as false alarms and missed alarms). Once this characterization is complete, the resulting Markov model (or, more generally, semi-Markov model) can be used to derive the statistics of any relevant quantity that is dependent upon the status of the system. Among these are the reliability and availability of the system, but the statistics of other quantities such as the time to first passage of a particular system status or a performance measure dependent on the system state history can also be calculated.

Despite their obvious utility for fault-tolerant system performance analysis, these models suffer from one serious drawback that has considerably limited their use. That drawback is that they tend to be computationally intractable even for relatively simple fault-tolerant system architectures. This intractability is the result of a number of factors:

1. The number of states can be large, particularly for complex systems comprising many components. Essentially, there are as many states in the model as there are distinct combinations of failed and unfailed components and RM decision statuses for which the system remains operative. Even the exploitation of symmetry and similar component behavior to reduce the model order can still leave a very large number of states in the final model.
2. The transient behavior, not the steady state behavior, is of primary interest. Because the components are subject to failure, the steady state for nearly all fault-tolerant systems is complete failure. Even when recovery of components is possible, the steady state may not become established until more time has elapsed than the useful lifetime

of the system (see comment 4 below). In either case, the transient behavior becomes the behavior of interest and steady state analysis techniques do not apply. This is particularly unfortunate when the model is semi-Markov in nature because the transient analysis of such processes requires the evaluation of convolution quantities (integrals or sums, respectively, for continuous or discrete time models) that require massive amounts of computer memory and computation time.

3. The time horizons of interest are often very long in absolute terms, though they still remain short relative to the time required for the process to reach the steady state. Typically, a fault-tolerant system will be used for operating intervals that are a significant fraction of the expected lifetime of its most failure-prone components. This fraction seldom approaches unity because the redundancy level of these components required to satisfy any reasonable specification on the system reliability would drive the price of the system high enough to justify the use of fewer, more reliable (and therefore more expensive) components. On the other hand, extremely short operating times would yield a probability of failure for any component that is so low that the extra investment in fault-tolerance would not be justified by the small increase in reliability. In light of 2 above then, the transient behavior of a Markovian process must be examined over time horizons on the order of the mean time to failure of the most failure-prone component. Given the current emphasis on the manufacture of highly reliable components, these time horizons can be extremely long.

4. A time scale separation tends to exist between the component failure process and the RM decision process. Failures tend to occur only rarely and therefore tend to have large time durations between them.

RM decisions, however, must occur quickly following a failure and tend to occur very rapidly relative to failure events. This means that the Markovian model of the behavior of the system status exhibits "fast" modes and "slow" modes. This time scale separation provides the motivation for the behavioral decomposition methods that are currently being investigated by us and by other researchers in the field.

The goal of this research project is to develop a method that generates approximate solutions to the generalized Markov process models that characterize fault-tolerant system behavior without the use of excessive computer memory or computation time. The behavioral decomposition alluded to in Comment 4 above provides the basis for the approach. However, the nature of fault-tolerant system models is such that extensions to existing theory are necessary in order to exploit the decomposition approach. These extensions and the numerical verification of their validity are the primary results obtained from the work reported here.

1.2 Previous and Related Work

A number of researchers have addressed various aspects of the problem of approximating the behavior of finite state Markov processes with weak interactions between groups of states. The most recent work to appear on this subject is that of Coderch [1]. This paper is derived from [2], which contains an extensive description of previous work in the area. Much of the work preceding [1] applied only to limited classes of finite state Markov processes and, in particular, were not applicable to semi-Markov processes or to processes with purely transient states. In [1], a method is described by which continuous time, finite state, weakly coupled Markov processes without transient states can be decomposed into transition operators that are valid for increasingly longer time scales. The result is a sequence of operators that describe the transition behavior of the process at each time scale such that the multiple time scale solution for the process behavior converges to the actual process behavior asymptotically as the small parameter representing the weak interactions converges to zero. Unfortunately, the method does not apply to semi-Markov processes and it has not been extended to apply to discrete time processes. Furthermore, the method requires the solution of very complex linear algebra problems, such as the description of nullspaces of operators, in the generation of the operators that are valid at each time scale.

Currently, an effort is underway to extend the results of [1] to finite state Markov processes evolving in both discrete and continuous time that include special types of transient states (called "nonsplitting transient states" in [4]). Some preliminary results of this effort are described in [3]. Further results are expected soon [4]. It should be noted that the results in [3] and [4], like those in the previously cited references, currently are applicable only to Markov processes. It is expected that [4] will include some results on semi-Markov processes, but the limitations of these results remain to be seen.

It should also be noted that the methods of [3] and [4], like those in [1,2], generate a description of the behavior of the process in sequentially longer time scales. It is frequently the case in fault-tolerant system analysis that the behavior of interest occurs only in the first time scale. This observation, combined with the difficulty that the methods of [3] and [4] have in dealing with transient states and the current lack of results for semi-Markov processes, suggests that an alternative method for dealing with these processes is of interest.

Much of the work reported here is an extension of the work reported by Korolyuk, et. al. [5,6]. These results apply to finite state semi-Markov processes with weak interactions, where the continuous time case is treated in [5] and the discrete time case in [6]. The interactions between the states is weak in the sense that the transition behavior depends upon a small parameter ε such that when ε is zero the process decomposes into noninteracting classes of states. The form of the transition behavior assumed by [5,6] is that the transition probabilities within a class include terms that are independent of ε while the interclass transition probabilities are all at least first order in ε. Also, it is assumed that the holding time densities associated with all transitions become compressed near the origin as ε becomes small. Finally, it is assumed that the decomposed classes that result from setting $\varepsilon=0$ are all ergodic. When all of these conditions are satisfied, it is shown in [5,6] that the behavior of the original process over time horizons on the order of t/ε can be approximated by a reduced order Markov process representing the interclass behavior in this time scale expanded by the stationary distribution of probability within a class that results from the ergodicity of each class when ε is zero. The parameters of the reduced order Markov process are expressed in terms of the transition probabilities of the original process and the mean holding times associated with the holding time distributions.

The results in [5,6] are very powerful for approximating the behavior of semi-Markov processes that satisfy all of the conditions in the first order time scale. Unfortunately, most models of fault-tolerant system behavior do <u>not</u> satisfy these conditions. This observation provides the motivation for much of the work to be reported here.

In particular, fault-tolerant system models tend to have two characteristics that violate the conditions imposed on the process by [5,6]. One is that the holding time densities do not compress as the small parameter representing the weak interclass interactions is made smaller. The reason for this is that the holding time densities for fault-tolerant system models are determined by the probability mass functions of the time needed for various sequential fault diagnosis tests to reach decisions. The behavior of the fault diagnosis tests typically occurs in the "fast" time scale, but it is not altered by changes in the failure rate of the components, which is usually the source of the small interaction parameters in these models. This situation is illustrated clearly by the model derived in Chapter 3 of [7], which is the 9-state model referred to in [8]. None of the holding time densities for this model display the explicit dependence on the scaled time t/ε that [5,6] assume (see Appendix C of [7]).

The other manner in which fault-tolerant system models often violate the conditions assumed in [5,6] is with respect to the ergodicity of the classes when $\varepsilon=0$. Many fault-tolerant systems include RM logic that shuts off a component permanently once it has been diagnosed as failed. If this diagnosis is the result of a false alarm, the corresponding system status state involves no failures and hence tends to be in the same class upon decomposition of the model as other no-failure states such as the state where no failures and no RM decisions have yet taken place. But the false alarm state in this case is a trapping state for this class when the failure probability (and hence ε) is set to zero. Therefore, this class is nonergodic. This tends to be true of many of the classes of states associated with models of fault-tolerant system behavior when irreversible RM logic is used by the system.

The work that was reported in [8] last year discussed some of the alternatives that were being investigated for circumventing the problems associated with applying the results of [5,6] to fault tolerant system models. In [8], it was noted that the ergodicity of the classes is actually a stronger condition than what is sufficient for the proofs presented in [5,6] to hold. In particular, it is sufficient that the inverse operator $[I - P_k + \pi_k]^{-1}$ exist where P_k and π_k are operators that are associated with the k^{th} class defined in [8,p. 7]. This observation leads to the interesting but not very useful conclusion that the results of [5,6] can be extended to models for which the weaker condition is satisfied by each class.

It was also reported in [8] that work had begun on circumventing the problem that the holding time densities for fault tolerant system models are not dependent on the small parameter representing the weak interactions. The approach described in [8] was to introduce a second small parameter that represented time scaling into the model. The holding time densities then took the appropriate form for application of the results of [5,6] provided the time scaling parameter was proportional to the original small interaction parameter. It was speculated that the time-scaled results would exhibit the asymptotic convergence to the correct behavior implied by the results of [5,6]. Work had just begun on investigating this speculative hypothesis for continuous time models.

1.3 Research Goals for the Year

The goals for the year of effort reported here were as follows:

1. Continue the extension of the results of [5,6] to models evolving in continuous time where the holding time densities do not depend directly upon the small interaction parameter ε but rather on a small time scaling parameter related to ε.
2. Conduct further investigations on nonergodic models by examining a number of continuous time examples. Attempt to identify a theoretical result regarding such models.
3. Develop results similar to [5,6] as extended by the two previous goals for discrete time semi-Markov models of fault tolerant systems.
4. Develop a means for generating the exact solution to models of simple fault-tolerant systems for the purpose of comparison with the results generated by the approximate technique.

The next section of this report will discuss the progress made on these goals during the past year.

II. PROGRESS SUMMARY

In this section, the work of the past year is summarized and is related to the goals that were discussed above. Numerous references are made to [7], which is the S.M. thesis of Siu-Kwong Chu that was completed under the support of this grant. This thesis is included as Appendix A of this report for easy reference.

2.1 Time-scaling of Continuous Time Models

In [8], the idea was put forward that when the time axis over which a semi-Markov model of fault-tolerant system behavior evolves is scaled by a small parameter δ, the holding time densities in the model take the form that is required for the application of the asymptotic theorems of [5,6] provided the parameter δ is proportional to ε. This idea is explained rigorously in section 2.2.1 of [7]. After introducing this time scaling, it is possible to rederive the results that are of interest for asymptotic approximations to the behavior of these semi-Markov models.

Let E be the state space of a finite state semi-Markov process that evolves in continuous time t. Suppose that the process is observed with respect to the scaled time t/δ. Suppose further that the transition operator of the process is such that its (j,i) element representing transitions from state i to state j has the form:

$$P^{\varepsilon}_{ji}(t') = p^{\varepsilon}_{ji}\, F_{ji}(t'/\delta) \qquad i,j \in E$$

where t' represents scaled time and where the eventual transition probabilities p^{ε}_{ji} take the form:

$$p^{\varepsilon}_{ji} = \begin{cases} p^{(k)}_{ji} - \varepsilon\, q^{(k)}_{ji} & i,j \in E_k \\ \varepsilon\, q^{(k)}_{ji} & i \in E_k,\ j \in E_k \end{cases}$$

Here it is assumed that the state space E decomposes into weakly interacting classes $\{E_1, E_2, \ldots, E_n\}$. It is also assumed that the $p^{(k)}_{ji}$ for each E_K sum to unity, hence when $\varepsilon=0$ the classes E_i become noninteracting and each describes a valid semi-Markov process.

Now let $\tau^{(i)}_{rk}$ be the sojourn time (in scaled time) of the process in class E_k when it begins from state $i \in E_k$ and transits to class E_r. Let $\phi^{(i)}_{rk}(s)$ denote the characteristic function of $\tau^{(i)}_{rk}$. Then, if the $p^{(k)}_{ji}$ for each k represent the transition probabilities of an ergodic Markov chain, then the $\phi_{rk}^{(i)}(s)$ are independent of the superscript i and they take the form:

$$\phi_{rk}(s) = \frac{\sum_{i\epsilon E_k} \pi_i^{(k)} \sum_{j\epsilon E_r} q_{ji}^{(k)}}{\sum_{i\epsilon E_k} \pi_i^{(k)} \sum_{j\epsilon E_k} (\frac{\delta}{\epsilon} s\ a_{ji}\ p_{ji}^{(k)} + \epsilon\ q_{ji}^{(k)})}$$

$$= p_{rk} \frac{\Lambda_k/\alpha}{\Lambda_k/\alpha + s}$$

where the $\pi_i^{(k)}$ are the stationary probabilities of the ergodic semi-Markov process associated with class E_k and the a_{ji} are the mean holding times associated with the $F_{ji}(t)$ in the original time scale. The quantities in the second expression above are defined in [7, sec. 2.2.2]. Note that this expression takes the form of the characteristic function of a Markov process transition operator with eventual transition probability p_{rk} and transition rate time constant Λ_k/α. Thus, the interclass transitions are Markovian in scaled time.

The proof of this result can be found in [7, sec. 2.2.2].

The derivation of the result expressed above makes possible the analysis of continuous time semi-Markov models of fault tolerant system behavior <u>provided the model has ergodic classes</u> (note the underlined condition above). Many fault tolerant system models violate this condition, as was discussed in the Introduction. However, many fault-tolerant systems that do not employ irreversible fault isolation logic do produce models with ergodic classes. Therefore, this result is a positive step toward analysis of models for these types of systems.

The manner in which the result above can be used for such analyses is as follows. Suppose a model for a fault tolerant system has been constructed and one is interested in calculating the state probabilities for the model at some relatively large value of time t in order to assess the reliability (or some other status-related property) of the system. Suppose further that the model satisfies the conditions stated in the result above. Then the approximate class occupancy probabilities at the desired time can be calculated by scaling time appropriately, constructing the Markov process that approximately governs interclass behavior from the result above (this is called the enlarged process in [7]) and solving this relatively easy Markov process problem. It is assumed here that the initial condition is known for the state probabilities and therefore also for the class occupancy probabilities. The results should be rescaled back to the original time scale. Then, finally, the approximate state probabilities can be evaluated by weighting the stationary probability distribution associated with each class when $\varepsilon=0$ by the appropriate approximate class occupancy probability.

The derivation of the result above and the construction of the approximate evaluation method discussed in the preceding paragraph complete the work necessary to satisfy Goal 1.

To illustrate the approximate evaluation procedure, a model for a generic fault tolerant system was constructed and solved using both "brute force" numerical convolution techniques and the approximate technique described above. The system consisted of three components where at least one unfailed component must be available for the system to remain operating. It was assumed that the failure diagnosis algorithm used sequential tests in combination with logic that is described in detail in sec. 3.1 of [7]. The tests were assumed to have second order Erlang distributions for their times to decision. The logic included the possibility of recovering components that have previously been diagnosed as failed, thereby leading to a model that has ergodic classes. The complete model is described in secs. 3.3 through 3.5 and Appendix C of [7]. The model has 9 states which decompose into three classes when the small failure rate is set to zero.

The exact state probability histories are obtained numerically and are described in chapter 4 of [7]. It should be noted that a very large amount of computational effort was required to generate these exact solutions. The approximate model is also constructed and solved in chapter 4 of [7]. The approximate solutions were, for the most part, obtained with just the aid of a hand calculator. Only when complete time histories were desired was it necessary to resort to the use of a computer. Upon comparison of the results, one finds that the largest error in the evaluation of any of the state probabilities by the approximate method for this example is less than 1% of the value obtained by numerical means (which itself is subject to a small amount of error) for times greater than the longest mean holding time of the sequential tests, where the assumed mean time between failures is 3 orders of magnitude longer than this.

These results are very encouraging, but they are not sufficient to conclude that the approximate technique always works so well. In order to further investigate the properties of the approximate technique with the time scaling included, a number of four-state semi-Markov models were examined. These models were chosen to reflect various characteristics that larger fault tolerant system models tend to possess. By keeping the dimension at 4, however, it is possible to generate the true behavior of the model with relative ease whereas models of larger dimension are extremely difficult to solve (recall the comments above regarding the nine-state model). Even four-state models are difficult enough to solve, however, that symbolic manipulation was necessary to generate the exact solutions. This is true despite the fact that none of the holding time densities in the models were assumed to be any more difficult than second order Erlang.

The five cases of four-state models that were examined are discussed in detail in chapter 5 of [7]. The approximate method produced very accurate results in every case that was examined. The comparison between the results was almost always exact to 4 decimal places except in the very early time periods before the startup transient of the process has decayed.

One of the cases of four-state models that was examined was a model that did not have ergodic classes (Case IV). The fact that the approximate technique still produced extremely accurate results suggested that we investigate further the ergodicity condition and its impact on the results from which the approximate method is derived. The work accomplished in this area is described in the next section.

2.2 Relaxation of Ergodicity Condition

Many fault tolerant systems yield generalized Markovian models of their behavior that decompose into classes that satisfy all of the conditions for applying the approximate technique except the condition that they be ergodic when $\epsilon=0$. This is typically the result of irreversible logic structures in the RM algorithm for the system such that diagnostic decisions alone can permanently eliminate a component from use.

However, in the analysis of four-state models discussed above, it was noted that excellent results were obtained when the approximate method was applied to a case where the model did not possess ergodic classes. A single example is not sufficient to prove any statement regarding the applicability of the approximate method to models with nonergodic classes. However, these results did motivate us to examine the underlying reason that the method worked for this particular example.

The result of this investigation is the following theorem regarding models with nonergodic classes:

Theorem 1: Let a semi-Markov process depend upon ε such that it can be decomposed in the manner described in section 2.1. Suppose in addition that the imbedded Markov process transition operator P_k associated with the k^{th} class when $\varepsilon=0$ satisfies:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P_k^m = [v \ v \ \cdots \ v]$$

where v is a constant vector, *for every k*. Then the interclass transition behavior approaches the same enlarged Markov process behavior that was described in section 2.1 as ε approaches zero.

The proof of this theorem appears in chapter 6 of [7].

Theorem 1 considerably widens the class of semi-Markov models to which the approximate technique can be applied because the condition stated in the theorem is weaker than the ergodicity of the classes that was required by the previous results. Many fault tolerant system models possess the properties stated in the conditions of Theorem 1.

The analysis leading to Theorem 1 led us to consider the specific situations in which the conditions of the theorem are satisfied. This investigation led to the following refinement of the theorem:

Theorem 2: Let a semi-Markov process depend on ε such that it can be decomposed into classes as prescribed in section 2.1. The transition operator P_k of the imbedded Markov process associated with the k^{th} class when $\varepsilon=0$ will satisfy the condition of Theorem 1 if:

1. The k^{th} class is ergodic, or
2. P_k has one and only one eigenvalue of unity.

The proof of this theorem also appears in chapter 6 of [7].

It should be emphasized that Theorem 1 is still only a sufficient condition for the approximate technique to yield accurate results as the small parameter ε becomes small. In other words, there may exist semi-Markov models that do not satisfy these conditions whose behavior can still be approximated well by the approximate method. Theorem 2 provides a more restrictive but more easily checked sufficient condition.

Some examples of models that do and do not satisfy the sufficient conditions of Theorem 1 are presented in chapter 6 of [7]. One example in particular that does not satisfy the conditions includes a class that contains multiple trapping states when $\varepsilon=0$. We have begun an effort to extend the results to this case as well by searching for conditions under which the approximate method succeeds in approximating the interclass behavior.

The derivation of the two theorems discussed above represents our progress thusfar on Goal 2.

2.3 Discrete Time Models

All of the results described so far in this report have applied to continuous time models of fault tolerant system behavior. However, because the RM algorithm for the system is usually implemented on a digital computer with a significant time delay between successive applications of the diagnosis tests, fault tolerant system models are often purely discrete time in nature. Efforts have been made during the past year to derive results for discrete time processes that mimic those discussed above for continuous time processes. This section reports on these efforts.

Much of the work that has been accomplished this year for discrete time models has related to the adaptation of Korolyuk's limit theorem for semi-Markov processes [5] to semi-Markov chains. In addition, a limit theorem with time scaling for semi-Markov chains was also developed. The theorem statements are summarized below.

An important result that will be referred to in both theorems discussed is presented in Lemma 3.

LEMMA 3: Let $P^{(k)} = [p_{ji}^{(k)}]$ represent an imbedded Markov chain operator of a semi Markov chain E_k. Consider the system of equations below:

$$\phi_{rk}^{(i)}(z) - \sum_{j \in E_k} \phi_{rk}^{(j)}(z)\, p_{ji}^{(k)} = 0$$

The solution of the system of equations is independent of the superscript, that is:

$$\phi_{rk}(m) = \phi_{rk}^{(i)}(m) \quad \forall \quad i \in E_k$$

if and only if the imbedded Markov chain operator represented by the transition probability matrix $\{p_{ji}^{(k)} | i,j \in E_k\}$ has at most a single unit magnitude eigenvalue.

Thus, any ergodic imbedded Markov chain operator (for which all eigenvalues have less than unit magnitude) will satisfy Lemma 1. In addition, any <u>monodesmic</u> imbedded Markov chain operator (one that has only one trapping or absorbing state, and hence a single unit magnitude eigenvalue) will also satisfy Lemma 1. This assertion is similar to Theorem 2 for continuous time models.

The following theorem describes how a semi-Markov chain which is dependent on a small parameter ε can be approximately described by a Markov chain. This theorem is derived based on the results for semi-Markov processes in [5].

Semi-Markov chains are characterized by a finite set of states and by a distribution of the <u>holding time</u> or <u>sojourn time</u> in each state that is arbitrary for each state to which a transition can occur. A semi-Markov chain specializes to a Markov chain when the holding times for each state are identically exponentially distributed. The semi-Markov chains here are assumed to depend on a small parameter ε such that the state space can be decomposed into disjoint classes of states where the probabilities of departure from each class tend to zero along with ε. In addition, the total sojourn in each class is assumed to have a non-degenerate distribution in the limit as $\varepsilon \to 0$.

<u>THEOREM 4:</u> A Limit Theorem for Semi-Markov Chains

Let the set E of states of the semi-Markov chain be expressible as a sum of disjoint classes

$$E = \sum_{k=1}^{N^e} E_k \qquad k \in \{M \mid k = 1,2,\ldots N^e\} \tag{2.1}$$

Let $\gamma_{rk}^{(i)}$ be the sojourn of the semi-Markov chain in class E_k when it starts from state i and moves to class E_r. The following two conditions are assumed to hold

1. The elements of the <u>core matrix sequence</u> $\{g_{ji}^{\varepsilon}(m) | i, j\varepsilon E\}$ specifying the semi-Markov chain depend as follows on the small parameter ε:

$$g_{ji}^{\varepsilon}(m) = p_{ji}^{\varepsilon} h_{ji}(\tfrac{m}{\varepsilon}) \tag{2.2}$$

and where $h_{ji}(0) = 0$. The p_{ji}^{ε} may be expanded in a Taylor series about $\varepsilon = 0$. Taking only linear terms in ε:

$$\begin{aligned} p_{ji}^{\varepsilon} &= p_{ji}^{(k)} - \varepsilon\, q_{ji}^{(k)} + \ldots + O(\varepsilon); \quad i,j \in E_k \\ &= \varepsilon\, q_{ji}^{(k)} + \ldots + O(\varepsilon); \quad i \in E_k \ ; \ j \notin E_k \end{aligned} \tag{2.3}$$

The imbedded Markov chain obeys the usual Markov chain properties:

$$\sum_{j\varepsilon E_k} p_{ji}^{(k)} = 1; \text{ and } p_{ji}^{(k)} \in [0,1]; \ \forall\ i,j \in E_k; \ \forall\ k \in M \tag{2.4}$$

and

2. The imbedded Markov chain defined by the transition probability matrices $\{p_{ji}^{(k)} | i, jE\varepsilon_k \ \forall\ k\varepsilon M\}$ are ergodic with stationary probabilities $\{\pi_i^{(k)} | i\varepsilon E_k \ \forall\ k\varepsilon M\}$.

Then:

$$\lim_{\varepsilon \to 0} \Pr\{\Upsilon_{rk}^{(i)} \leq t\} = \gamma_{rk} [1 - \exp(-\Lambda_k t/T)] \tag{2.5}$$

where:

$$\gamma_{rk} = \frac{\sum_{i\varepsilon E_k} \pi_i^{(k)} q_i^{(kr)}}{\sum_{i\varepsilon E_k} \pi_i^{(k)} q_i^{(k)}} ,$$

$$\Lambda_k = \frac{\sum\limits_{i \in E_k} \pi_i^{(k)} \, q_i^{(k)}}{\sum\limits_{i \in E_k} \pi_i^{(k)} \, a_i^{(k)}} ,$$

Here:

$$q_i^{(rk)} = \sum_{j \in E_r} q_{ji}^{(k)} ,$$

$$q_i^{(k)} = \sum_{j \in E_r} q_{ji}^{(k)} ,$$

$$a_i^{(k)} = \sum_{j \in E_r} p_{ji}^{(k)} \bar{\gamma}_{ji} ,$$

$$\bar{\gamma}_{ji} = \sum_{m=0}^{\infty} m \, h_{ji}(m) .$$

Although the above theorem is useful, it is not directly applicable to most fault tolerant system models for two reasons: (1) the imbedded Markov chains for such models are usually non-ergodic, and (2) the holding time density functions are usually not dependent on m/ε but only on m. Hence, a necessary adjustment that must be made in the above theorem is to determine what conditions must be satisfied by the imbedded Markov chain (thus Lemma 3) and to incorporate time scaling into Theorem 4.

THEOREM 5: A Limit Theorem With Time Scaling for Semi-Markov Chains

Let the set E of states of the semi-Markov process be expressible as a sum of disjoint classes

$$E = \sum_{k=1}^{N^e} E_k \qquad 1 \in \{M \mid k = 1,2,\ldots N^e\}$$

Let $\gamma_{rk}^{(i)}$ be the sojourn of the semi-Markov chain in class E_k when it starts from state i and moves to class E_r. Let the following two conditions hold for the semi-Markov chain:

1. The elements of the <u>core matrix sequence</u> $\{g_{ji}^{\varepsilon}(m) | i,j\varepsilon E\}$ specifying the semi-Markov chain depend as follows on the small parameter δ:

$$g_{ji}^{\varepsilon}(m) = p_{ji}^{\varepsilon}\, h_{ji}(\tfrac{m}{\delta}) \tag{2.2}$$

and where $h_{ji}(0) = 0$. The p_{ji}^{ε} may be expanded in a Taylor series about $\varepsilon = 0$. Taking only linear terms in ε:

$$\begin{aligned} p_{ji}^{\varepsilon} &= p_{ji}^{(k)} - \varepsilon\, q_{ji}^{(k)} + \ldots + O(\varepsilon); \quad i,j \in E_k \\ &= \varepsilon\, q_{ji}^{(k)} + \ldots + O(\varepsilon); \quad i \in E_k\ ; \quad j \notin E_k \end{aligned} \tag{2.3}$$

The imbedded Markov chain obeys the usual Markov chain properties:

$$\sum_{j\varepsilon E_k} p_{ji}^{(k)} = 1; \text{ and } p_{ji}^{(k)} \in [0,1];\ \forall\ i,j \in E_k;\ \forall\ k \in M \tag{2.4}$$

and

2. The imbedded Markov chains defined by the transition probability matrices $\{p_{ji}^{(k)} | i,j\varepsilon E_k\ \forall\ k\varepsilon M\}$ have at most a single unit magnitude eigenvalue (hence, ergodic or monodesmic) with stationary probabilities $\{\pi_i^{(k)} | i\varepsilon E_k\ \forall\ k\varepsilon M\}$.

Then:

$$\lim_{\varepsilon\to 0} \Pr\{\gamma_{rk} \le t\} = \gamma_{rk}\,[1 - \exp(-\Lambda_k t/\alpha T)] \tag{2.5}$$

where γ_{rk}, Λ_k, $q_i^{(rk)}$, $q_i^{(k)}$, and $a_i^{(k)}$, were all defined in Theorem 4 and α is defined below.

$$\alpha = \frac{\delta}{\varepsilon} .$$

The results of Theorem 5 are being applied to examples of fault tolerant control systems for which semi-Markov chain reliability models have been derived. Three simple reliability models have been developed to date. The first is for a simple component monitoring system. A single non-essential component has a sequential test monitoring faults for the information of the pilot. This produces a 3-state model that can be decomposed into two classes. The second model is of a single-component dual redundant (SCDR) system. This model has six states and three non-ergodic classes. When a false alarm recovery test is incorporated into the second system, a model with nine states and three ergodic classes results.

These three models will be analyzed by applying the results of Theorem 5. The probabilities of occupying each class will be computed and will be compared to a numerical or analytical computation of the same quantities.

This work and its continuation represents our progress so far on Goal 3.

2.4 Generation of Exact Results

When approximate answers are derived to problems for which it is difficult or impossible to generate the exact answer, a question arises regarding the means by which these approximate answers can be validated. Obviously, it is the intent of the problem-solver to avoid the difficult procedure of generating an exact answer. Yet, without the exact answer, how can one be certain that the approximate answer is accurate? We face that dilemma here in calculating our approximate answers to fault tolerant system model behavior.

In section 2.1, we limited our consideration of model structures to four-state models with Erlangian holding times so that we could generate the exact answers relatively easily. In fact, as we discussed in section 2.1, it was still necessary to use a symbolic manipulation program to derive the true results because the numerical calculations were cumbersome.

Discrete time models of fault tolerant system behavior tend to be just as cumbersome. In selecting the three models of fault tolerant system behavior to analyze, we have been careful to choose simple ones. This allows us to analyze their behavior analytically before applying the approximate technique.

In this regard, our efforts have been directed toward using a symbolic manipulation package (MACSYMA) to obtain, in closed form, the z-transform solution to the discrete time models (that is, an expression for the state occupancy probability vector). From the analytical solution, a truncated Taylor series expression in ε can be found that can be compared with the results of applying Theorem 5. This will provide an expression for the first truncated term of the Taylor series and thus will provide an error bound on the approximation for these models.

In the proof of Theorem 5, all order ε^2 terms in the total probability equation are ignored. The resulting expression contains a zero and first order ε term. The zero order term is shown to vanish in the limit as ε approaches zero. The remaining first order ε term is left and ε may be cancelled, leaving the Theorem 5 result. However, a first order perturbation of the Theorem 3 result can be obtained by expanding the total probability equation to second order in ε and ignoring order ε^3 terms.

Again, the zero order term vanishes in the limit. With the remaining terms, $\phi_{rk}(z)$ is found, but the new expression contains terms proportional to ε. Including this perturbation term in ε should improve the numerical results that can be obtained for the class occupancy probabilities. This will be discussed in future progress reports.

This constitutes the progress we have made so far on Goal 4.

III. PAPERS AND PRESENTATIONS

No papers were derived from this work during this year. However, a paper is in progress based upon the work reported in sections 2.1 and 2.2 that will be submitted to an archival journal, probably Mathematics of Operations Research. Also, one S.M. thesis was completed this year, namely that of Siu-Kwong Chu. This thesis [7] is included here as Appendix A.

A presentation on this work and other fault tolerant system evaluation work was given by Prof. Walker at NASA-Langley Research Center in March. In addition, Prof. Walker has been invited to speak as part of an aerospace systems workshop at the American Control Conference in Seattle in June.

IV. PROJECTIONS FOR THIRD YEAR OF WORK

During the third year of work, the goals of the program are those that were stated in the renewal proposal. These are:

1. Investigate the possible further weakening of the conditions sufficient for the validity of the approximate results for continuous time models beyond the Theorems of section 2.2. Our primary emphasis here will be continuous time models for which at least one of the classes of the nonperturbed process contains more than one trapping state.

2. Continue the derivation of analogous results for purely discrete parameter models.

3. Complete the symbolic derivation of analytical solutions for the three models described in section 2.4. Use the results to find either an alternative form for the discrete time approximate results or an error bound in terms of ε on the approximate results. Generalize the error bound, if possible.

4. Use the sampled Monte Carlo techniques of [9] to generate valid "truth" results with which the approximate results can be compared.

V. FINANCIAL AND MANPOWER STATUS

The manpower complement remained unchanged from the proposal. Professor Bruce K. Walker continues as the Project Director, devoting approximately 20% of his academic year time and 60% of his summer time to the project. The two graduate students, Siu-Kwong Chu and Norman M. Wereley, continue as full-time graduate Research Assistants supported by the project. Margaret McCabe provides clerical assistance. No changes are anticipated from the manpower arrangement proposed in the renewal proposal.

The financial aspects of the project have also followed the proposal closely with one exception. The cost underrun from the first year was added to the second year budget, partly as capital equipment funds. Air Force approval was given for this change by Capt. Dwight McGhee in a letter dated 11 December 1985. The capital equipment money was used to purchase an IBM Personal Computer Model AT, which is now the primary means of computation and wordprocessing for all three participants in the grant.

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APPROXIMATE BEHAVIOR OF
GENERALIZED MARKOVIAN MODELS OF
FAULT-TOLERANT SYSTEMS

by

Siu Kwong Chu

S.M. Thesis

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APPROXIMATE BEHAVIOR OF GENERALIZED MARKOVIAN MODELS OF FAULT-TOLERANT SYSTEMS

by

Siu Kwong Chu

Submitted to the Department of Aeronautics and Astronautics on January 23rd., 1986 in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE IN AERONAUTICS AND ASTRONAUTICS.

Abstract

Problems associated with the evaluating state probability histories of large state space models of fault-tolerant system are explained, and it appears that Korolyuk's Limit Theorem for semi-Markov processes may be a solution to these problems that approximates the aggregated original semi-Markov process by a reduced order Markov process. The Theorem is modified and extended to apply to approximate fault-tolerant system models in a new time scale. The approximate technique is then developed by expanding the approximate Markov process state probability histories with the stationary probability distributions associated with the aggregated groups of states of the original semi-Markov process. The technique is demonstrated with a realistic 9-state model and five 4-state models which mimic the class to class transition structure of typical fault-tolerant system models, and the results show that accurate approximation is achieved for these examples after a short transient period. In addition, the ergodicity sufficient condition imposed on the semi-Markov process to be approximated is relaxed. As a result fault-tolerant system models with certain types of non-ergodic classes can also be solved by the approximate technique.

Thesis Supervisor: Professor Bruce K. Walker
Title: Assistant Professor of Aeronautics and Astronautics

In Memory of My Father,

Chu Pak Wai

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I wish to express my sincere and grateful thanks to my thesis supervisor, Professor Bruce K. Walker, for his valuable advice and guidance. Without his insight into this project, I may still be working on $\phi_{rk}(s)$ at the time of submitting this thesis. I also thank him for his thorough review of the thesis draft that significantly added to its clarity.

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Finally, I thank my mother and other family members, whose support has always been my greatest asset.

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Notation

E	state space of semi-Markov process
E_k	k-th partition or class of state space of a semi-Markov process
$F_{ji}(\cdot)$	cumulative probability density function for time to transition from state i to state j
$h_{ji}(\cdot)$	holding time probability density function for transitions from state i to state j
$H(\cdot)$	holding time probability density function matrix
p_{ji}^{ϵ}	eventual transition probability from state i to state j of perturbed semi-Markov process
$p_{ji}^{(k)}$	eventual transition probability from state i to state j in class k of non-perturbed semi-Markov process
p_{rk}	eventual transition probability from aggregated "state" k to aggregated "state" r of the approximate Markov process
P	eventual transition probability matrix
$P(\cdot)$	transition kernel matrix
$P_{E_i}(\cdot)$	total probability in class i of perturbed semi-Markov process
$P_{ji}^{\epsilon}(\cdot)$	kernel element for transition from state i to state j of perturbed semi-Markov process
$>W(\cdot)$	waiting time greater than $(\cdot)$

δ	time scaling factor
ϵ	small parameter representing the constant failure rate
$\phi_{rk}(\cdot)$	transition kernel for class k to class r transitions of the approximate Markov process
$\Phi(\cdot)$	interval transition probability matrix
λ_0	parameter for false alarm decision time probability density function
λ_1	parameter for isolation decision time probability density function
$\lambda_{W0}/\lambda_{W1}$	parameters for failed/unfailed indication decision time probability density function of the self-test given the component is working
$\lambda_{F0}/\lambda_{F1}$	parameters for failed/unfailed indication decision time probability density function of the self-test given the component is failed
Λ_k	constant transition rate out of aggregated "state" k of the appproximate Markov process
$\pi_i(\cdot)$	probability in state i of semi-Markov process
$\pi^{(k)}(\cdot)$	total probability in class k of the original semi-Markov process
$\pi_k^e(\cdot)$	probability in state k of the approximate Markov process (or enlarged process) i.e., approximate total probability for class k of the original semi-Markov process
$\pi_i^{(k)}$	stationary probability in state i which belongs to class k of the non-perturbed semi-Markov process
$\pi_{M_i}^{(k)}$	stationary probability in state i of the imbedded non-perturbed Markov process for class k of the semi-Markov process

$\bar{\tau}_{ji}$	mean holding time of transition from state i to state j
$\bar{\tau}_{i}$	mean holding time in state i without regard to the destination
$\bar{\tau}$	mean holding time of a semi-Markov process
$\tau_{rk}^{(i)}$	the sojourn random variable of the semi-Markov process in E_k when it starts from state i, $i \in E_k$ and moves to E_r

Nomenclature

BITE	Built-In Test Equipment
CDF	Cumulative Distribution Function
FDI	Fault Detection and Isolation
MTTF	Mean Time To Failure
PDF	Probability Density Function
PMF	Probability Mass Function
RM	Redundancy Management
SL	System Loss
SPRT	Sequential Probability Ratio Test
VSST	Vector Shiryayev Sequential Test

Chapter 1
Introduction

1.1 Background

A *fault-tolerant control system* is a system designed with redundant capacity to perform its mission. That is, it can do its job using more than one configuration of its components, e.g. sensors and actuators and information processing capability. The on-line detection and isolation of failed components and the reconfiguration of the system's architecture is performed by the system's *Redundancy Management* (RM) scheme. The fault-tolerant approach enhances system reliability and performance. There are many application areas where ultra-high system reliability is necessary or desirable. One such area is the control of nuclear power plants where the consequences of improper control system behavior may be serious indeed. There are space missions for which the desired operational lifetime of the spacecraft is many years. The air traffic control system and many military systems are also subject to very high reliability requirements. There is also a desire for increased reliability in computerized banking systems, chemical process control systems, medical monitoring systems, transportation systems, and many more. As a result, growing attention is being given to the design of components for long life, to quality control during manufacture, and testing and maintenance policies which enhance reliable system operation. Despite these efforts to improve the reliability of individual components, the resulting system reliability is still often inadequate for some reliability requirements. As a result, there is increasing interest in fault-tolerant system designs which allow components to fail but still provide a means for the system to continue to function.

The growing use of fault-tolerant system designs has in turn spurred interest in methods for assessing the reliability and performance of such systems. The traditional methods of reliability evaluation are based on combinatorial analysis of combinations of component failures [7] . They generally consider only the probabilistic occurrences of component failures and seldom account for the probabilistic nature of the outcomes of any on-line monitoring test that might be used by the fault-tolerant system in an effort to detect and identify such failures and to reconfigure the system to remove from use any failed components. In addition, classical reliability analysis produces as its sole result the probability that the system will maintain its integrity over the duration of its operating time. No information is provided on the performance of the system during the transient period of the mission.

Since classical reliability analysis fails to quantify fault-tolerant system time behavior, other alternatives must be considered. Naturally, in this age of the high-power main-frame computer, Monte Carlo simulation is one option. This method consists of building, with a computer program, a probabilistic model of the system under investigation. If the system of interest is properly modeled for various random effects that bear on it and sufficient simulation runs are obtained, then essentially any aspect of the system performance can be statistically evaluated from the simulations. However, as is pointed out in [10], the drawback of Monte Carlo technique stems from the fact that a sufficient number of simulations must be available. For a system with a component failure rate as low as 10^{-9} per sec., the number of simulations needed to generate statistically significant results about failures must exceed one billion. Furthermore, the fault-tolerant system to be simulated is frequently rather complex, often involving multiple instruments and a hierarchical architecture for the *Failure Detection and Isolation* (FDI) logic. Consequently, obtaining reliable results by Monte Carlo technique is often prohibitively costly in terms of the required computational effort.

The use of Markov chain theory [9, 6] has shown promise as a means for evaluating the performance of those fault-tolerant systems which employ FDI tests that are of the single sample variety, that is, the information that is used for FDI is gathered and discarded at each time sample. However, single sample FDI tests generally have a relatively high likelihood of decision errors, particularly in noisy signal environments. In such situations, fault-tolerant systems are always equipped with digital computers that execute FDI tests based on several samples of the monitoring data at each time sample. Such tests include moving window tests and tests of a completely sequential nature. Such tests are not memoryless. Therefore, the systems in which they are employed are not conducive to the compact treatment by the application of Markov chain analysis that is possible for systems employing only single sample tests.

The Markov modeling technique mentioned in the previous paragraph must be generalized in order to capture the non -memoryless nature of the sequential RM strategy employed in many fault-tolerant systems. More specifically, the model must account for the time delays associated with processing a sequence of observations before a FDI decision is made. Some effort has been made to analyze such systems and it appears that the *generalized Markovian* (or *semi-Markov*) modeling methods [10, 8] are applicable to some systems of this type. In addition to the necessary assumptions, a problem with this reliability evaluation method is that the large number of states in the model causes the computation of results to involve excessive amounts of computer storage and computation time. (Usually, each state in a generalized finite-state semi-Markovian model of fault-tolerant system behavior represents a particular combination of specific component failure modes and of RM decisions.) The reason for this for both continuous time and discrete time models is as follows: For quantitative continuous time system performance analysis, the state probability distribution $\pi(t)$ at every time t must be evaluated. With known $\pi(0)$, standard time-invariant semi-Markov theory yields,

$$\underline{\pi}(t) = \Phi(t)\underline{\pi}(0) \tag{1.1}$$

where Φ(t), interval transition probability matrix, is the solution of the following matrix convolution integral equation (see appendix A.2 for the details of the derivation, and the notations),

$$\Phi(t) = \; > W(t) + \int_0^t d\tau\, \Phi(t-\tau)[P \,\square\, H(\tau)], \qquad \Phi(0) = I \tag{1.2}$$

The above equation is in a form that can be solved analytically by the Laplace transform technique. It is not difficult to obtain Φ(t) in closed form for systems that comprise only two or three states. However, for complex systems with a large number of states (for example, the model for a dual-redundant engine controller has 30 states [2] flight control system models will have many more), it will become intractable to obtain a closed form solution even with the help of symbolic manipulation software, e.g. MACSYMA. The reason is as follows: Solving Eq. (1.2) for Φ(t) involves the problem of inverting an N x N matrix symbolically, where N is the number of states of the system model. Unlike the case in numerical analysis where the number of operations required for a matrix inversion is on the order of N^3, in symbolic inversion the number of operations for a N x N matrix whose elements are as simple as a single term function of s is on the order of N!. It should also be pointed out that a symbolic operation is also more complicated than its counterpart in numerical operations, which is usually a floating point multiplication. In addition, the computer memory required for storing intermediate expressions is extremely large. So the problems associated with memory storage and computation time prohibit the use of a symbolic manipulation program in solving for $\Phi(t)$ analytically for a continuous time model. On the other hand, for discrete-time semi-Markov models, π_k, the state probability distribution at time step k with known $\pi(0)$, can be expressed as,

$$\underline{\pi}(k) = \Phi(k)\underline{\pi}(0) \tag{1.3}$$

where $\Phi(k)$ is recursively generated by (see appendix A.1 for the details of derivation and notations),

$$\Phi(k) = {}^{>}W(k) + \sum_{m=0}^{k} \Phi(k-m)[P \square H(m)], \quad \Phi(0) = I \tag{1.4}$$

It can be seen that a convolution sum is involved. This implies that for a system with N states, approximately $2kN^2$ values must be stored in order to compute $\Phi(k)$ and hence $\pi(k)$. For $N = 20$ and $k = 100{,}000$ as might be the case for a simple flight control system operating with RM updates at a rate of 50Hz for 35 minutes, the storage required is approximately 80×10^6 values or 640 megabytes of storage for accurate single precision state probability distribution calculations. The number of floating point multiplications required for calculating $\Phi(100{,}000)$ is approximately 7×10^{12}. This poses the same problem as the continuous time model. These computational and memory burden problems encountered in the reliability and performance analysis of complex fault-tolerant systems employing non-memoryless FDI tests provides the motivation for the work described in this thesis.

The goal of this work is to reduce the problems encountered in complex system reliability analysis by expanding upon the asymptotic approximation technique for semi-Markov processes described in [4, 5] and applying them to fault-tolerant system models. The idea, basically, is as follows: Consider a time-invariant finite-state continuous parameter semi-Markov process whose state probability distribution is given by $\pi(t)$ for $t \geq 0$ with $\pi(0)$ known. Then $\pi(t)$ can be evaluated according to Eq. (1.1) and (1.2). Suppose the process depends on a small parameter ϵ such that the state space of the process can be partitioned into disjoint classes $E_1, \ldots E_m$ when $\epsilon = 0$. That is, no classes can communicate with any of the other classes when ϵ is zero. Suppose further that the *Probability Density Functions* (PDFs) that govern the transitions between states also depend on ϵ in the "right" form (as will be explained in Chapter 2) and let $\underline{\pi}^e(t)$ be the probability distribution associated with this aggregated grouping of states. Then it can be shown [11] that $\underline{\pi}^e(t)$ evolves according to the Kolmororov backward equations governing a time-invariant Markov process, that $\pi_k^e(t) = \lim_{\epsilon \to 0} \sum_{i \in E_k} \pi_i(t/\epsilon)$ and that the parameters defining the Markov process can be derived from that of the original semi-Markov process. In less rigorous terms, this means that the long-term behavior of the original model, that is the distribution $\pi^{(k)}(t/\epsilon)$ after it is aggregated, is asymptotically well-approximated by the distribution $\pi_{(k)}^e(t)$ which evolves as a Markov process with known transition behavior as the small parameter ϵ nears zero. If a stationary probability distribution $\underline{\pi}^{(k)}$ exists for each disjoint class of states E_k, then the approximation for the probability in state i is [11],

$$\pi_i(t) \approx \pi_i^{(k)} \pi_k^e(\epsilon t) \tag{1.5}$$

As can be seen, the approximate technique involves two elements, namely the stationary probability distribution and the *approximate Markov process* (or *enlarged process*). These results are also applicable to discrete-time time-invariant finite state semi-Markov models [12].

The application of these results to the reduction of the complexity of the reliability evaluations based upon generalized Markovian models is reasonably straight-forward if the system model has all the characteristics mentioned above. Most fault-tolerant systems produce generalized Markovian models that are approximately in the form necessary to apply these results because the rarity of the component failure events relative to the rate at which RM decisions are typically made yields the small parameter ϵ which must be present in the characterization. A problem typically arises with the form of the state to state transition holding time density functions, this problem, however, will be dealt with in this thesis.

1.2 Organization of Thesis

The mathematical tool that is used to model fault-tolerant systems is the theory of semi-Markov processes. They are very similar to Markov processes but with one more degree of freedom that make them well suited for capturing the random delay behavior of RM decisions for nonmemoryless tests. Asymptotic enlarging of semi-Markov processes [4, 5] is the primary tool that is used to accomplish the goal of this thesis. However, general fault-tolerant systems yield semi-Markov models whose state to state transitions do not behave the same as that described in the references there. Therefore, the theory will be extended here to apply to typical fault-tolerant system models and the parameters for the resulting approximate Markov processes will be derived in Chapter 2.

In Chapter 3, the structure of an example fault-tolerant system is described and the assumptions used in the model construction for it are stated. After defining all the system states, a 9-state transition kernel matrix is constructed which completely characterizes the system behavior. It is shown that the system model can be decomposed into three classes of states when the component failure rate is equal to zero. The transition kernel is then decomposed into the standard form that will be used in the subsequent chapter to calculate the parameters of the approximate Markov process.

Chapter 4 deals with the analysis of accuracy of the two elements of the approximation technique. That is, the evolution of the aggregated state probability distribution calculated by the semi-Markov approach is compared with state probability distribution of the enlarged process and the normalized probability distribution is compared with the stationary probability distribution in each class.

The enlarged process approximation method is further tested in Chapter 5 with a general 4-state semi-Markov model. Five different cases are presented which capture five different possible class to class transition types that might typically occur in a fault-tolerant system model.

The sufficient condition imposed on the semi-Markov processes for the approximate technique to be applied is relaxed and two theorems associated with this relaxation are established in Chapter 6.

Some limitations of the enlarged process approximation approach are examined in Chapter 7.

Chapter 8 concludes the thesis with a discussion of the work and its contributions and suggestions for the directions that further research might take.

Chapter 2
Theory of Enlarged Semi-Markov Processes

As it is pointed out in the Introduction, the mathematical tools used in this thesis are classical semi-Markov process theory and the theory of enlarged semi-Markov processes. Semi-Markov process theory is used to model the probabilistic behavior of a fault-tolerant system. The resulting mathematical model of a complicated fault-tolerant system with a large number of components and several different levels of RM decisions is a high dimensional model with a large transition kernel matrix. Usually, it is impractical to obtain the desired state probability distribution history over the mission length due to limited computer memory storage and the high computational cost. The enlarged semi-Markov process theory, to be described in Section 2.1, is used to approximate the large dimension semi-Markov process by a low dimension Markov process, which characterizes the evolution of probability among groups of states. That is, each state of the enlarged process represents a group of states of the original semi-Markov process. Frequently in fault-tolerant system models, each enlarged process state represents a group of states from the original model with the same number of working components but having different RM configurations. However, enlarged semi-Markov process theory as it appears in the current literature does not apply to fault-tolerant system models, as will be explained in Section 2.2. Therefore, the theory will be extended in Section 2.2.2 in order to apply it to fault-tolerant system models.

2.1 Korolyuk's Limit Theorem for Semi-Markov Processes

References [3,4] describe the conditions under which a perturbed semi-Markov process can be approximated over long time frames by a Markov chain. There are essentially two of these conditions. First, the kernel of the semi-Markov process must depend on a small positive parameter ϵ in such a way that the entire space of states of the semi-Markov process E can be split into disjoint classes of states $E = \sum_{k=1}^{m} E_k$, where the probabilities of departure from each class and of the sojourn time in a given state both tend to zero with ϵ . The total sojourn time in each class is assumed to have a nondegenerate distribution in the limit as $\epsilon \rightarrow 0$ (when ϵ=0, the process will be referred to the *non-perturbed* semi-Markov process while the original process will be referred to as the *perturbed* semi-Markov process). Mathematically this condition can be expressed by the following equations,

$$P^{\epsilon}_{ji}(t) = p^{\epsilon}_{ji} F_{ji}(t/\epsilon), \qquad i,j \in E; \tag{2.1}$$

$$p^{\epsilon}_{ji} = \begin{cases} p^{(k)}_{ji} - \epsilon q^{(k)}_{ji} & i,j \in E_k, \\ \epsilon q^{(k)}_{ji} & i \in E_k,\ j \notin E_k, \end{cases} \tag{2.2}$$

$$\text{where} \sum_{j \in E_k} p^{(k)}_{ji} = 1, \qquad i \in E_k, \quad 1 \leq k \leq m.$$

where p_{ji} is the eventual transition probability of the original process from state i to state j, $F_{ji}(t/\epsilon)$ is the *Cumulative Distribution Function* (CDF) of the holding time for transitions from state i to state j.

Second, the Markov chains defined by the transition probability matrices $p^{(k)}_{ij}(i,j \in E_k, \quad 1 < k < m)$, must be ergodic with stationary probability distributions $\pi^{(k)}_i$ $(i \in E_k, 1 < k < m)$ When these conditions are satisfied by a perturbed semi-Markov process, then its behavior can be approximated by a

Markov chain. More specifically, if $\tau_{rk}^{(i)}$ is the sojourn of the semi-Markov process in class E_k when it begins from state i and moves to class E_r , then [4] shows that the cumulative distribution function of the random variable can be expressed by an exponential function when ϵ becomes vanishingly small:

$$\lim_{\epsilon \to 0} P\{ \tau_{rk}^{(i)} < t \} = p_{rk} (1 - e^{-\Lambda_k t}) \tag{2.3}$$

As can be seen from the above equation, the dependence on i disappears on the right hand side of the equation. That is, each state in class E_k has the same exponential holding time density function for transitions to class E_r for all r. So all the states in class E_k can be merged together and the aggregated model has the characteristic of a Markov process.

The second part of condition 1, defined by Eq. (2.2), is often satisfied by a fault-tolerant system model. If the system components all have small constant failure rates proportional to ϵ, then each class of states for the enlarged process can be formed by grouping together all the states that have the same groups of working and failed components but with different statuses of the RM logic. The class-to-class transitions are then possible only through the small possibility of failure of a component. When ϵ=0, i.e. when no failures can take place, the only transitions that are possible are those within each class due to the outcomes of the RM decisions. If there is Built-In Test Equipment (BITE) included in the RM system, a component that was previously isolated as failed by the RM can be brought back on line. For this kind of system, the imbedded Markov chain for each class is generally ergodic. Then the second part of condition 1 is satisfied. The remaining condition that has to be satisfied is defined by Eq. (2.1) or the first part of condition 1. Usually, this condition is **not** satisfied by a fault-tolerant system model. The reason is as follows: if ϵ is small, i.e. the *Mean Time To Failure*

(MTTF) of the components is large, say hundreds of hours, then the holding time of the transition, particularly those within a class, is determined only by the noise in the signals and the threshold set by the FDI test designer. So, as the failure rate tends to zero, the RM decision delay will not be affected by the failure rate. So, the transition kernel of a fault-tolerant system semi-Markov process model will not take on the form implied by Eq. (2.1). Because Eq. (2.1) is not satisfied, the enlarged process, if it can even be formulated, may be an invalid approximation to the aggregated semi-Markov process model.

2.2 Extension of Korolyuk's Work

As described in Section 2.1, the only condition in Korolyuk's theorem that is not satisfied by fault-tolerant system is that the FDI decision delay, and therefore the holding time probability density functions, does not depend on the small parameter ϵ. However, the state transition delay of a semi-Markov process would be dependent on a small parameter mathematically if the temporal line on which the delay was originally measured is scaled, say by a time scaling factor δ. In this way, fault-tolerant system models can be modified to satisfy all the conditions required for the enlarged process results to be applied. Section 2.2.1 shows how the transition kernels of a semi-Markov process depend on the small parameter δ when the process is characterized on a new temporal line. Section 2.2.2 will derive the parameters of the Markov process that approximates the behavior of the aggregated, time scaled semi-Markov model.

2.2.1 Changing the Time scale of a Perturbed Semi-Markov Process

A fault-tolerant system model with a finite number of states evolving in continuous time is a semi-Markov model which is completely characterized by its transition kernel matrix. The standard form of the (i,j) element of the matrix is as follows:

$$P_{ji}(t) = p_{ji}\, h_{ji}(t) \tag{2.4}$$

where p_{ji} is the eventual transition probability and $h_{ji}(t)$ is the conditional transition time probability density function for transitions from state i to state j. The eventual transition probability is the probability that the process that entered state i on its last transition will enter state j on its next transition. Before making this transition, the process "holds" for a random time in state i, where the time is governed by the conditional transition time probability density function. In fault-tolerant systems with small component failure rates of order ϵ, $h_{ji}(t)$ is related to the PDFs of the time delay of the FDI tests. Obviously, $h_{ji}(t)$ does not in general depend on ϵ. However it will depend on another small parameter δ, the time scaling factor, if the original temporal line on which the FDI decision delay was measured is scaled. If a stochastic process is observed in another time scale that is $1/\delta$ times that of the original, then the holding time PDF $h_{ji}(t)$ certainly will be affected but the p_{ji} will be the same because the eventual transition probability p_{ji} only characterizes the transition probability from state i to state j for the next transition whenever it occurs. Therefore, it is not related to the time scale in which the process is observed. However, the "new" holding time PDF is not obtained by just replacing the argument of the original PDF by t/δ because if the original $h_{ji}(t)$ is replaced by $h_{ji}(t/\delta)$ for the change of time scale, then integration of $h_{ji}(t/\delta)$ from time $t=0$ to time $t=\infty$ does not produce 1. This means that $h_{ji}(t/\delta)$

is not a proper holding time PDF. For a change of time scale of a stochastic process, the CDF $F_{ji}(t)$ of the corresponding PDF $h_{ji}(t)$ must be found, and the argument of the CDF, t, must be replaced by t/δ. Then the holding time PDF of the process observed in the new time scale is $h'_{ji}(t') = \frac{d}{dt'} F_{ji}(t'/\delta)$. So, the statistics of the process in the new time scale depend on the small parameter δ, the time scaling factor. If δ equals ϵ, i.e. if the time scaling factor is equal to the failure rate of the components, then the condition is satisfied. But δ is not necessarily equal to ϵ for the derivation of the enlarged process and the enlarged process will be derived in the next section.

2.2.2 Derivation of $\phi_{rk}(s)$ of a Time-Scaled Perturbed Process

As pointed out in the last section, a time-scaled version of the original process is not required in the evaluation of the approximate solution. So what follows is the proof that the aggregated semi-Markov process in scaled time evolves as a Markov process and the derivation of the parameters of the Markov process. A similar approach to that of reference [4] will be used in this section for the proof and the derivation of the parameters.

It is assumed that the system semi-Markov model depends on the small failure rate parameter ϵ in such a way that the entire space of states of the model E can be split into disjoint classes of states $E=\{E_1....E_k\}$ such that the probabilities of departure from each class tend to zero as ϵ tends to zero. In addition, if the process is observed on a temporal line $1/\delta$ times that of the original then the sojourn in a given state tends to zero as δ tends to zero. To illustrate this point, consider a process that is observed in terms of hours while it originally was described in terms of seconds. Then the PDF describing the delay for transitions from state i to state j will be "crushed" near the origin, so the sojourn in state i will be small in units of hours. Because the whole process is observed in a new time scale, all of the $F_{ji}(\ )$ will depend on the small parameter δ.

A time-scaled semi-Markov process with the above characteristics can be characterized by the following equations,

$$p^{\epsilon}_{ji} = \begin{cases} p^{(k)}_{ji} - \epsilon q^{(k)}_{ji} & i,j \in E_k, \\ \epsilon q^{(k)}_{ji} & i \in E_k,\ j \notin E_k, \end{cases} \tag{2.5}$$

where p^{ϵ}_{ji} is the eventual transition probabilities of the imbedded Markov chain, and the non-perturbeded eventual transition probabilites $p^{(k)}_{ji}$ satisfy the following equation

$$\sum_{j \in E_k} p^{(k)}_{ji} = 1 \qquad i \in E_k, \quad 1 \leq k \leq m \tag{2.6}$$

and the element of the transition probability matrix can be expressed as,

$$P^{\epsilon}_{ji}(t) = p^{\epsilon}_{ji} F_{ji}(t/\delta) \quad i,j \in E \tag{2.7}$$

where $F_{ji}(\cdot)$ is the CDF of the transition delay for the process in the original time scale. Eq.(2.7) is a generalization of the form of the transition probability matrix elements that define the semi-Markov process.

If $\tau^{(i)}_{rk}$ denotes the sojourn of the semi-Markov process in class E_k when it starts from state i and moves to E_r and $\delta\varsigma_{ji}$ denotes the sojourn of the semi-Markov process in state i, with the CDF $F_{ji}(t)$, while δ^{ϵ}_{ji} are the indicators of transition from state i to the state j, so the $E\{\delta^{\epsilon}_{ji}\}=p^{\epsilon}_{ji}$, then the random quantities $\tau^{(i)}_{rk}$ can be obtained by using the expression for the total probability :

$$P\{\tau_{rk}^{(i)} \le t\} = \sum_{j \in E_k} P\{\delta_{ji}^{\epsilon} = 1, \quad \delta\varsigma_{ji} + \tau_{rk}^{(j)} \le t\}$$
$$+ \sum_{j \in E_r} P\{\delta_{ji}^{\epsilon} = 1, \quad \delta\varsigma_{ji} \le t\} \tag{2.8}$$

Hence

$$P\{\tau_{rk}^{(i)} \le t\} = \sum_{j \in E_k} \int_0^t P\{\tau_{rk}^{(j)} \le t-u\}\, dP_{ji}^{\epsilon}(u) + \sum_{j \in E_r} P_{ji}^{\epsilon}(t) \tag{2.9}$$

Using the Laplace transform,

$$\phi_{rk}^{(i)}(s) = E\{e^{-s\tau_{rk}^{(i)}}\} \tag{2.10}$$

$$p_{ji}^{\epsilon}(s) = \int_0^{\infty} e^{-st}\, dP_{ji}^{\epsilon}(t) \tag{2.11}$$

then eq.(2.9) becomes,

$$\phi_{rk}^{(i)}(s) = \sum_{j \in E_k} \phi_{rk}^{(j)}(s)\, p_{ji}^{\epsilon}(s) + \sum_{j \in E_r} p_{ji}^{\epsilon}(s) \tag{2.12}$$

Combining the Laplace transform of Eq. (2.7) and Eq. (2.5) :

$$p_{ji}^{\epsilon}(s) = (p_{ji}^{(k)} - \epsilon q_{ji}^{(k)})(1 - \delta s a_{ij} + 0(\epsilon)), \qquad j \in E_k \tag{2.13}$$

$$p_{ji}^{\epsilon}(s) = \delta q_{ji}^{(k)} + 0(\epsilon), \qquad j \notin E_k \tag{2.14}$$

substituting these expressions in eq.(2.12), it becomes,

$$\phi_{rk}^{(i)}(s) - \sum_{j \in E_k} \phi_{rk}^{(j)}(s)\, p_{ji}^{(k)} = - \sum_{k \in E_r} (\, \delta s a_{ji}\, p_{ji}^{(k)} + \epsilon q_{ji}^{(k)}\,)\, \phi_{rk}^{(j)}(s)$$

$$+ \epsilon \sum_{j \in E_r} q_{ji}^{(k)} + 0(\epsilon\, \delta) \tag{2.15}$$

Passing to the limit as ϵ and $\delta \rightarrow 0$, the functions $\phi_{rk}^{(i)}(s)$ are found to satisfy the system of equations,

$$\phi_{rk}^{(i)}(s) - \sum_{j \in E_k} p_{ji}^{(k)}\, \phi_{rk}^{(j)}(s) = 0 \tag{2.16}$$

It follows from this and the assumption that the imbedded Markov chain defined by the transition probabilities $p_{ji}^{(k)}$ (i, j $\in E_k$) is ergodic, that (see [1]) the solution of system Eq. (2.16) is independent of the superscript, i.e. for all $i \in E_k$, $\phi_{rk}^{(i)}(s)=\phi_{rk}(s)$. Multiplying Eq. (2.15) by the stationary probabilities $\pi_i^{(k)}$ and summing over all $i \in E_k$, then cancelling ϵ, the following is obtained,

$$\sum_{i \in E_k} \pi_i^{(k)} \sum_{j \in E_k} (\, \tfrac{\delta}{\epsilon}\, s\, a_{ji}\, p_{ji}^{(k)} + \epsilon q_{ji}^{(k)}\,)\, \phi_{rk}^{(j)}(s) = \sum_{i \in E_k} \pi_i^{(k)} \sum_{j \in E_r} q_{ji}^{(k)} \tag{2.17}$$

or,

$$\phi_{rk}(s) = \frac{\displaystyle\sum_{i \in E_k} \pi_i^{(k)} \sum_{j \in E_r} q_{ji}^{(k)}}{\displaystyle\sum_{i \in E_k} \pi_i^{(k)} \sum_{j \in E_k} (\, \tfrac{\delta}{\epsilon}\, s\, a_{ji}\, p_{ji}^{(k)} + \epsilon q_{ji}^{(k)}\,)} \tag{2.18}$$

or,

$$\phi_{rk}(s) = p_{rk} \frac{\Lambda_k/\alpha}{\Lambda_k/\alpha + s} \tag{2.19}$$

where

$$\alpha = \frac{\delta}{\epsilon} \tag{2.20}$$

$$p_{kr} = \frac{\sum_{i \in E_k} \pi_i^{(k)} q_i^{(rk)}}{\sum_{i \in E_k} \pi_i^{(k)} q_i^{(k)}} \tag{2.21}$$

$$A_k = \frac{\sum_{j \in E_k} \pi_i^{(k)} q_i^{(k)}}{\sum_{j \in E_k} \pi_i^{(k)} a_i^{(rk)}} \tag{2.22}$$

$$q_i^{(rk)} = \sum_{j \in E_r} q_{ji}^{(k)} \tag{2.23}$$

$$q_i^{(k)} = \sum_{j \in E_k} q_{ji}^{(k)} \tag{2.24}$$

$$a_i^{(k)} = \sum_{j \in E_k} a_{ji} p_{ji}^{(k)} \tag{2.25}$$

$$a_{ji} = \int_0^\infty t \, dF_{ji}(t) \tag{2.26}$$

This completes the proof that any semi-Markov model with the properties stated above can be approximated by a Markov process whose parameters were also derived. The Markov process evolves in a longer time scale, i.e. $1/\delta$ times that of the original process. For instance, if δ=1/3600 and if the original semi-Markov model evolves in seconds then the approximate enlarged Markov process will evolve in hours.

One of the sufficient conditions in the derivation in this Chapter for the enlarged process is that all the classes must be ergodic. This condition is not generally satisfied by all fault-tolerant system models. One non-ergodic model will be examined in Chapter 5 and this issue will also be discussed in Chapter 6.

There are two parameters involved in the derivation, namely ϵ and δ, but the parameter that actually affects the behavior of the original semi-Markov process is ϵ while δ is just a time scaling factor that relates the time scale of the approximate Markov process and that of the original semi-Markov process. However, there is no known way to show how small ϵ must be for the Markov process to be a good description of the behavior of the aggregated semi-Markov model. So, assessment of the effect of the small parameters will have to rely on empirical results. For this purpose, a fault-tolerant system semi-Markov model will be constructed in the next chapter.

Chapter 3
Construction of Fault-Tolerant System Model

In the preceding chapter it was proved that under certain conditions such as vanishingly small ϵ and ergodic classes, an aggregated, perturbed, time-scaled semi-Markov process evolves asymptotically as a Markov process and the parameters of the approximate Markov process were also derived. However, bounds on the size of ϵ are not known for the Markov process to be a good approximation of the original semi-Markov process. As mentioned before, ϵ is usually the system component failure rate. Then the question arises: For the approximation to be reasonably good, would ϵ have to be extremely small ? In another words, do the MTTFs of the flight control system components of subsystems have to be unrealistically big, say 5 years, which is equivalent to ϵ=4.47x10^{-9}, for the aggregated system model to behave approximately as a Markov process ? This provides the motivation for the construction of a generalized Markovian fault-tolerant system model in this chapter for such investigation and for the demonstration of the approximation technique. Since the base-line numerical results of the model will be calculated from semi-Markov theory the system model will have to be small enough to avoid excessive memory storage and computational burden, but it will be rich enough to include sequential FDI tests and self-tests that are found in many fault-tolerant systems. Since the theory developed in Chapter 2 is in the continuous time domain, the model will also be formulated in continuous time. Any conclusions obtained in continuous time theory should also be valid in the discrete time case.

This chapter begins with a section that describes the architecture and FDI structure of an example fault-tolerant system. The next section states the assumptions that are made in the model constructuon. The state definitions will be presented in Section 3.3. The formation of the transition kernel of the semi-Markov process is illustrated in the next section. Decomposition of the transition kernel into the required form is included in the following section.

3.1 Structure of Three-Component Fault-Tolerant System

Suppose that the fault-tolerant system, or subsystem, comprises three independent instruments which are measuring (or actuating or otherwise operating on) a single scalar quantity. Such situations arise in such applications as flight control, (e.g. body rate sensors along a given axis and actuators for segmented control surfaces), highly reliable data processors, (e.g. redundant synchronizing clocks). In the measuring instruments case, three independent observations of a scalar quantity are available. With a set of two linearly independent *parity equations*, those three independent observations are used to generate a *vector parity residual sequence*. The RM in the system relies on the *Vector Shiryayev Sequential Test* (VSST) which makes use of the vector parity residual sequence to detect and identify the failure mode (see Section 3.1.3 of [10]). In contrast to other sequential tests (e.g. the *Sequential Probability Ratio Test* or SPRT), there is no need with the VSST for a separate isolation stage once a failure mode is detected. When an instrument is identified as failed, it is removed from the system by the reconfiguration scheme.

Once an instrument is removed from the system, a SPRT self-monitoring test is initiated on the isolated instrument. The intent here is to model the implementation of BITE monitoring that is often included in real systems. The self-test produces either a failed or an unfailed indication on an isolated instrument and when there are two consecutive indications that the instrument is unfailed, then the instrument is brought back on line and the VSST FDI test is reinitiated. It is assumed that no effort is made to detect further failures when two unflagged instruments remain available.

3.2 Assumptions in Model Construction

The complete structure of the fault-tolerant system was described in the last section. Before we proceed to construct the associated generalized Markovian model several assumptions will be made. Some of these assumptions make this example, and most other fault-tolerant systems easier to analyze by semi-Markov technique. These assumptions are as follows:

(a) The time to failure in any particular instrument is exponentially distributed and independent of the status of other instruments and the RM decisions.

(b) The probability of more than one event occurring during any dt is negligible. These events include failures of components and decisions by the RM system or by the self-tests.

Assumptions (a) and (b) are widely used in the analysis of fault-tolerant system performance, so no further justification for them will be given here.

Following [10] , consider the situation where a failure occurs at some time other than a state transition time, that is it does not occur at a renewal time (where the VSST is reinitialized). In this case, VSST will have established values of the test statistics which are distributed according to distributions conditioned on the hypothesis that no failure is present. Since the VSST is initialized with zero test statistics, this implies that the test has a "head start" towards detection at the time of a failure which is likely to yield a smaller delay to detection relative to the delay associated with newly initialized test statistics. However, if the test has been designed to achieve a low false alarm probability, the effect of assuming the unaffected test statistics' is at its initial condition should be minor relative to the effect of making the same assumption for the test that is affected by the failure. Thus:

(c) For VSST and SPRT, the occurrence of a failure is assumed to coincide with a renewal time for the test.

The last assumption below is an unrealistic one. However, it can still capture the non-memoryless nature of the self-test:

(d) Failed and unfailed indications by the self-test are independent.

Although this assumption is not the case for a SPRT, under this assumption and assumption (b), the time to failed and unfailed decisions will have the same density function but with different eventual transition probabilities. If the failed indication rate is higher than the unfailed indication rate given the isolated component is failed, then there will be a higher eventual probability for failed indications that will appear in the transition kernel. These assumptions will be used in the transition kernel construction in Section 3.4.

3.3 State Definitions

We are now in a position to define the states for the semi-Markov model of the example fault-tolerant system described in Section 3.1. The state characterizations must include all the information necessary to formulate the transition kernels for the exit transitions out of each state. In this system, it is necessary to know the following in order to characterize each state:

1. The number of instruments that are available for use.

2. Of these, how many of them have failed.

3. If an instrument has been isolated by FDI as failed, the status of the isolated component and the number of unfailed indications by the self-test for this instrument.

Consider what happens if all of the possible system configurations are enumerated as the system states. For example, suppose that the condition where the first instrument is failed and the other two are working, in the case of 3 available instruments, is enumerated as state 1, the second instrument failed and the other two working, in the case of 3 available instruments, is enumerated state 2, etc. Then the resulting model will have twenty-six states. However, since all the instruments for this example are the same, there will be no difference between states 1 and 2 in terms of the number of failed and working instruments or in terms of how many failed instruments are still in use. Only the number of failed and working instruments and the number of unfailed indications from the self-test are necessary in the state definitions. So, by merging the states, the dimension of the model can be greatly reduced, in this case to just nine.

The unacceptably degraded condition, which is a trapping state, is denoted by SL (*system loss*) and is assumed to comprise all system configurations that involve two or more failed components.

Let the state characterizations be denoted by the following notation where brackets indicate sets of possibilities from which one and only one element will appear in each state characterization:

$3 / \begin{Bmatrix} 0 \\ F \end{Bmatrix}$ 3 instruments available for use

$2 / \begin{Bmatrix} 0 \\ F \end{Bmatrix} / \begin{Bmatrix} C \\ W \end{Bmatrix} / \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ 2 instruments available for use

In the case where 3 instruments are available for use, the leading 3 represents the three available instruments. In the second entry, 0 represents no failure is present, F indicates there is 1 failed component. The case of 2 failures is not included because it represents a system loss. When two instruments are available for use, the notation with the leading 2 follows the same convention as before and represents the two available instruments. The 0 or F in the second entry indicates the presence of no failure or 1 failure among the three components, respectively. C or W in the third entry indicates whether the isolated component has been correctly or wrongly isolated, respectively. The last entry represents the number of consecutive unfailed indications from the self-test for this instrument. As an example, consider the state denoted by 2/F/W/1. This means that two instruments are available for use and one of the three is failed. Furthermore, the isolated component has been wrongly isolated (i.e. it is not the failed one). Finally, there has been one unfailed indication from the self-test for the isolated component. An exhaustive list of all the states and the state numbering scheme follows:

state	s.c.n.[1]	state description
1	3/0	3 available, none failed, VSST in operation.
2	2/0/W/0	2 available, none of the three failed, no unfailed indication from self-test.
3	2/0/W/1	2 available, none of the three failed, 1 unfailed indication from self-test.
4	3/F	3 available, 1 failed, VSST operation (i.e. detection delayed.
5	2/F/C/0	2 available, 1 of the three failed and correctly isolated, no unfailed indication from self-test.
6	2/F/C/1	2 available, 1 of the three failed and correctly isolated, 1 unfailed indication from self-test.
7	2/F/W/0	2 available, 1 of the three failed but incorrectly isolated, no unfailed indication from self-test.
8	2/F/W/1	2 available, 1 of the three failed but incorrectly isolated, 1 unfailed indication from self-test.
9	SL	system loss.

As can be seen above, it requires 9 states to describe the operational states of this fault-tolerant system. From now on, this system model will be referred to as the 9-state model.

The transitions out of each of the nine states correspond to the occurrence of one of the random events such as component failures and RM decisions. The state transition event trees for all 9 states are given in Figure 3-1. For clarity of how the system can transit from one state to another state, a state transition diagram is

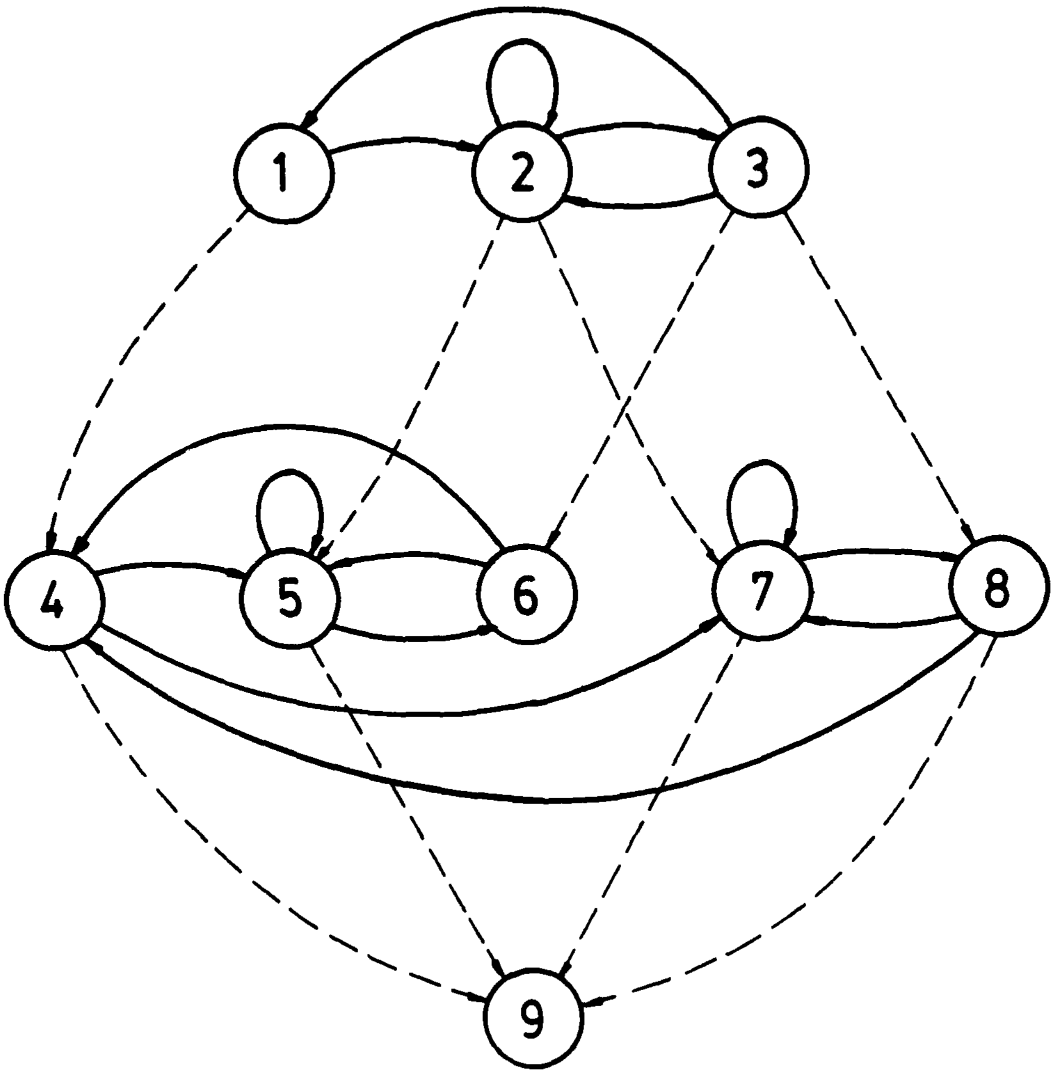
[1]state characteristic notation

		Transition due to occurrence of
1 3/0	2 2/0/W/0	false alarm by VSST.
	4 3/F	failure of one of the 3 instruments.
2 2/0/W/0	2 2/0/W/0	failed indication from self-test.
	3 2/0/W/1	unfailed indication.
	5 2/F/C/0	failure of isolated instrument.
	7 2/F/W/0	failure of one of the two available instruments.
3 2/0/W/1	1 3/0	2nd. consecutive unfailed indication from self-test, instrument brought back on line.
	2 2/0/W/0	failed indication from self-test.
	6 2/F/C/1	failure of isolated instrument.
	8 2/F/W/1	failure of one of the two available instruments.
4 3/F	5 2/F/C/0	correct isolation by VSST.
	7 2/F/W/0	wrong isolation by VSST.
	9 SL	2nd. failure of instrument.

Figure 3-1: State transition event trees

(5) 2/F/C/0	(5) 2/F/C/0	failed indication from self-test.
	(6) 2/F/C/1	unfailed indication after previous failed indication from self-test.
	(9) SL	2nd. failure of instrument.
(6) 2/F/C/1	(4) 3/F	2nd. consecutive unfailed indication from self-test, instrument brought on line.
	(5) 2/F/C/0	failed indication from self-test.
	(9) SL	2nd. failure of instrument.
(7) 2/F/W/0	(7) 2/F/W/0	failed indication from self-test.
	(8) 2/F/W/1	unfailed indication from self-test.
	(9) SL	2nd. failure of instrument.
(8) 2/F/W/1	(4) 3/F	2nd. consecutive unfailed indication from self-test, instrument brought on line.
	(7) 2/F/W/0	failed indication from self-test.
	(9) SL	2nd. failure of instrument.
(9) SL	(9) SL	trapping state.

Figure 3-1, continued



class 1

class 2

class 3

– – – no transition when ϵ=0

Figure 3-2: Transition diagram of the 9-state model shown in Figure 3-2.

3.4 Transition Kernel Matrix for the 9-State Model

In order to formulate the transition kernels for the 9-state model in closed form, the following conditional decision time density functions associated with the two sequential tests employed by the system and the time to failure density function of each instrument are assumed to be known:

$f_V^0 =$ density function of time to isolation by VSST under condition that no failure is present, with parameter λ_0.

$f_V^F =$ density function of time to isolation by VSST under condition that one failure is present, with parameter λ_1.

$f_S^{W0} =$ density function of time to failed indication by self-test SPRT under condition that no failure is present in the isolated instrument, with parameter λ_{W0}.

$f_S^{W1} =$ density function of time to unfailed indication by self-test SPRT under condition that no failure is present in the isolated instrument, with parameter λ_{W1}.

$f_S^{F0} =$ density function of time to failed indication by self-test SPRT under condition that a failure is present in the isolated instrument, with parameter λ_{F0}.

$f_S^{F1} =$ density function of time to unfailed indication by self-test SPRT under condition that a failure is present in the isolated instrument, with parameter λ_{F1}.

$f^F =$ density function of time to failure of each instrument, with parameter ϵ.

These decision time density functions for the tests will be assumed to be 2nd order Erlang functions (see Appendix B for the properties of the density function) and they are relatively realistic because the sequential tests are unlikely to reach their decision either a very short time or a very long time after they are initiated. Rather, they are more likely to reach a decision around a region of time that is some distance after the test is initiated. To illustrate this point, a Monte Carlo simulation[2] for the correct decision *Probability Mass Function* (PMF) of a VSST was obtained and is plotted in Figure 3-3. It shows that most of the decisions are reached at around 18 seconds after the test is initiated.

After the conditional decision time density functions for the tests are known, the transition kernel can be constructed by considering what the kernel elements actually represent:

$$\begin{aligned} P_{ji}(\tau)\, d\tau &= p_{ji}\, h_{ji}(\tau) d\tau \\ &= Pr\{\, i \rightarrow j \text{ in } [\, \tau, \tau + d\tau\,) \mid \text{enter } i \text{ at } 0 \,\} \end{aligned}$$

By expanding the meaning of $i \rightarrow j$ and the definition of conditional probability, $P_{ji}(\tau)d\tau$ can be rewritten in two different forms as follows:

$$\begin{aligned} P_{ji}(\tau)\, d\tau = Pr\{\, & i \rightarrow j \text{ in } [\, \tau, \tau + d\tau\,) \text{ and no } i \rightarrow k \text{ at any } t < \tau \\ & \text{for } k{=}1,2,\ldots\ldots,N \mid \text{enter } i \text{ at } 0 \,\} \end{aligned} \tag{3.1a}$$

$$\begin{aligned} = Pr\{\, & i \rightarrow j \text{ in } [\, \tau, \tau + d\,\tau\,) \mid \text{no } i \rightarrow k \text{ at any } t < \tau \\ & \text{for } k{=}1,2,\ldots\ldots,N \text{ and enter } i \text{ at } 0 \,\} \\ & \qquad Pr\{\, \text{no } i \rightarrow k \text{ at any } t < \tau \\ & \text{for } k{=}1\ldots\ldots,N \mid \text{enter } i \text{ at } 0 \,\} \end{aligned} \tag{3.1b}$$

Following [10], the form in Eq. (3-1a) will be called the *direct form* because it is simply a restatement of the definition of $P_{ji}(t)$. Eq. (3-1b) will be called the *conditional decomposition* of $P_{ji}(t)$. For clarity, Eq. (3.1b) can be modified as,

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[2] Monte Carlo simulation source code is supplied by the author of reference [10]

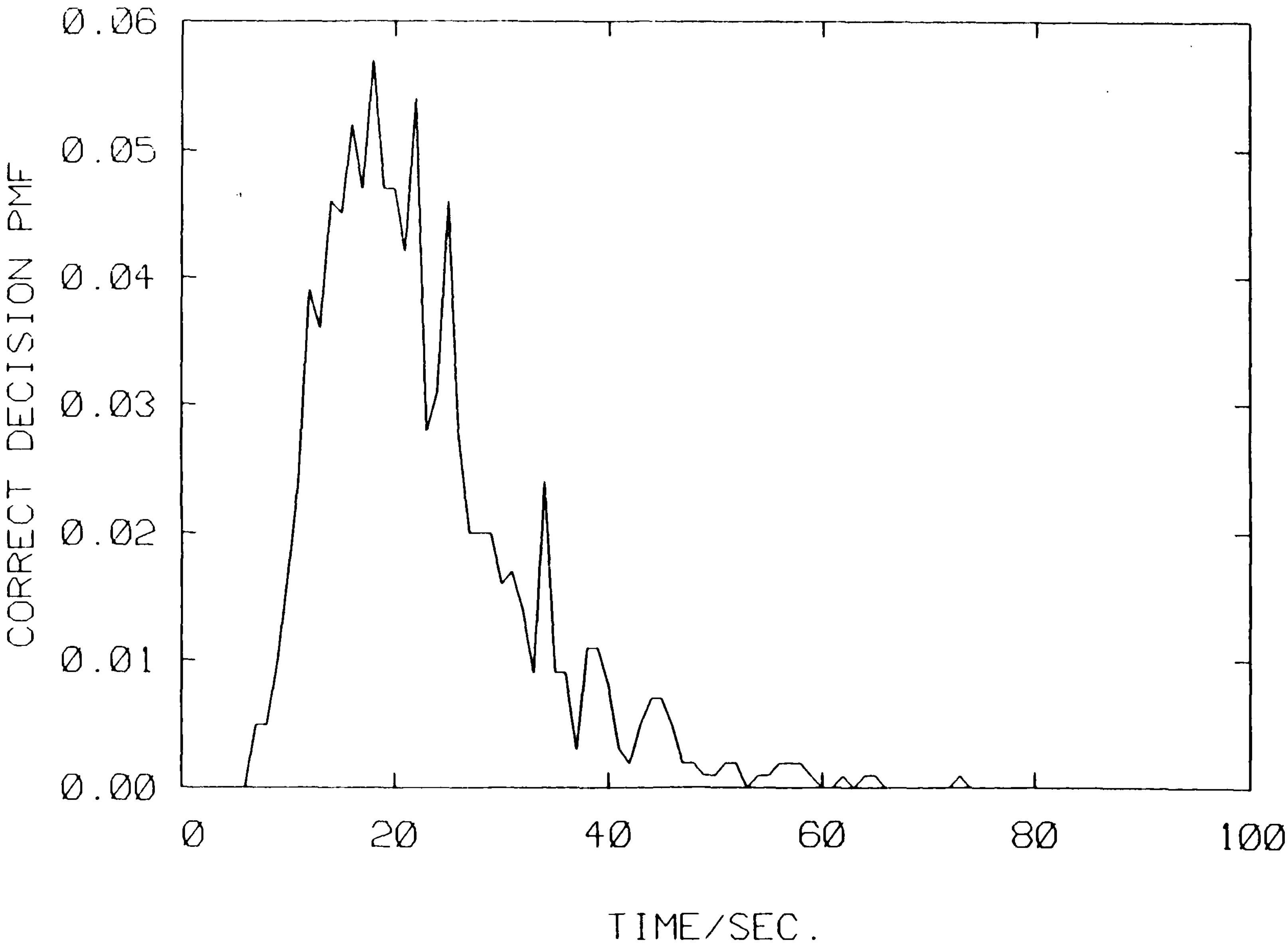

Figure 3-3: Correct decision PMF of VSST by Monte Carlo simulation

$$P_{ji}(\tau)\,d\tau = \frac{Pr\{\ i \rightarrow j \ in \ [\ \tau, \tau + d\tau\ ) \ and \ no \ i \rightarrow j \ at \ any \ t < \tau \mid \ no \ i \rightarrow k \ at \ any \ t < \tau \ for \ k{=}1,2,\ldots,j{-}1,j{+}1,\ldots,N \ and \ enter \ i \ at \ 0\ \}}{Pr\{\ no \ i \rightarrow k \ at \ any \ t < \tau \ for \ k{=}1,2,\ldots,j{-}1,j{+}1,\ldots,N \mid enter \ i \ at \ 0\ \}} \tag{3.1c}$$

The conditional decomposition of $P_{ji}(t)$ for each j and i provides a complete definition of the behavior of the semi-Markov process. Construction of the transition kernel by use of the conditional decomposition is particularly useful for fault-tolerant system models because the eventual transition probability for each state transition is generally not known.

The construction of two representative transition kernel elements of the 9-state model is described below. First, the transition kernel element $P_{21}(t)$ for transition from state 1 to state 2 is derived. State 1, which represents all the instruments are working, with state characterization notation 3/0, can only transit to state 2 with state characterization notation 2/0/0 and to state 4 with state characterization notation 3/F. Hence, the transition from 1 to 2 represents the occurrence of a false alarm by the VSST in the absence of a failure of any one of the instruments. Using the definition of Eq. (3-1b), the transition kernel element is derived as follows:

$$P_{21}(\tau)\,d\tau = \frac{Pr\{\ 1 \rightarrow 2 \ in \ [\ \tau, \tau + d\tau\ ) \ no \ 1 \rightarrow 2 \ at \ any \ t < \tau \mid \ no \ 1 \rightarrow 4 \ at \ any \ t < \tau \ and \ enter \ 1 \ at \ 0\ \}}{Pr\{\ no \ 1 \rightarrow 4 \ at \ any \ t < \tau \mid enter \ 1 \ at \ 0\ \}} \tag{3.2}$$

In terms of the conditional density functions of the test decision times defined at eq. (3.2a):

$$P_{21}(\tau)\,d\tau = f_v^0(\tau)d\tau\,[\ 1 - \int_0^{\tau} f^F(\mu)d\mu\ ]^3 \tag{3.3}$$

If

$$f_v^0 = \lambda_0^2 \, \tau \, e^{-\lambda_0 \tau} \quad \text{(2nd order Erlang)}$$

$$f^F = \epsilon e^{-\epsilon t} \quad \text{(exponential)}$$

then

$$P_{21}(\tau)\, d\tau = \lambda_0^2 \, \tau \, e^{-\lambda_0 \tau} \, d\tau \, [\, e^{-\epsilon t} \,]^3$$

or

$$P_{21}(t) = \lambda_0^2 \, t \, e^{-(\lambda_0 + 3\epsilon)\, t} \tag{3.4}$$

Another transition kernel element to be considered explicitly here is the one representing transitions from state 2 to state 5. State 2 and State 5 have state characterization notation 2/0/W/0 and 2/F/C/0, respectively. Other states that state 2 can transit to are states 2, 3 and 7 corresponding to state notation 2/0/W/0, 2/0/W/1 and 2/F/W/0, respectively. Hence the transition from state 2 to state 5 represents the occurrence of a failure in the isolated instrument in the absence of any failures among the two available instruments and of any decision reached by the self-test. Then the transition kernel element can be derived as follows:

$$\begin{aligned} P_{52}(\tau) = Pr\, \{ \, 2 \rightarrow 5 \ in \ [\, \tau, \tau + d\tau\,) \ and \ no \ 2 \rightarrow 5 \ \ at \ any \ t < \tau \,|\\ no \ 2 \rightarrow 2,3,7 \ at \ any \ t < \tau \ and \ enter \ 2 \ at \ 0 \, \} \\ Pr\, \{ \, no \ 2 \rightarrow 2,3,7 \ at \ any \ t < \tau \,|\, enter \ 2 \ at \ 0 \, \} \end{aligned} \tag{3.5}$$

with assumptions (a) and (d), Eq. (3-3a) can be rewritten in terms of the conditional density functions of the test decision times:

$$P_{52}(\tau)d\tau = f^{F}(\tau)d\tau \left[1 - \int_0^{\tau} f_s^{W0}(\mu)d\mu \right] \left[1 - \int_0^{\tau} f_s^{W1}(\mu)d\mu \right] \left[1 - \int_0^{\tau} f_s^{F1}(\mu)d\mu \right]^2 \tag{3.6}$$

By substituting expressions for the density fucntions:

$$P_{52}(\tau)d\tau = \epsilon e^{-\epsilon\tau}d\tau\,(\lambda_{W0}\tau + 1)\,e^{-\lambda_{W0}\tau}\,(\lambda_{W1}\tau + 1)\,e^{-\lambda_{W1}\tau}\,\epsilon e^{-2\epsilon\tau}$$

$$= \epsilon\,(\lambda_{W0}\tau + 1)\,(\lambda_{W1}\tau + 1)\,e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,\tau}\,d\tau$$

or

$$P_{52}(t) = \epsilon\,(\lambda_{W0}t + 1)\,(\lambda_{W1}t + 1)\,e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t} \tag{3.7}$$

Two of the twenty-six nonzero transition kernel elements were derived above. The remaining elements are included in Appendix C. The fault-tolerant system model is completely characterized by this transition kernel matrix, and state probability histories can be derived from it by using Eq. (1.2). Any aspect of the system performance statistics can be derived from it. The complete transition kernel matrix is given in Eq. (3.8).

If ϵ is set equal to zero in Eq. (3.8) , that is, if no failures can occur among the instruments, then the transition kernel will be reduced to the form shown in Figure 3-4. This matrix can be partitioned into a block diagonal matrix consisting of 3 blocks. This implies that no transitions occur between the states associated with different blocks. Then the states within each of these three blocks form a closed class. Therefore, when the original process is reduced to a non-perturbed semi-Markov process, the resulting process consists of 3 classes. The first class comprises states 1, 2 and 3, each of which has all three instruments working but with different RM levels. Class two contains all the system states with exactly one

$$
P(t)=\begin{bmatrix}
\cdot & \cdot & \text{[illegible]} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \text{[illegible]} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\text{[illegible]} & \cdot & \cdot & \cdot & \cdot & \text{[illegible]} & \cdot & \text{[illegible]} & \cdot \\
\cdot & \text{[illegible]} & \cdot & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \text{[illegible]} & \cdot & \text{[illegible]} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \text{[illegible]} & \cdot & \text{[illegible]} & \cdot & \cdot & \text{[illegible]} & \text{[illegible]} & \cdot \\
\cdot & \cdot & \text{[illegible]} & \cdot & \cdot & \cdot & \text{[illegible]} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \text{[illegible]} & \cdot
\end{bmatrix}
$$

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Eq. (3.8)

$$P(t)=\begin{bmatrix} 0 & 0 & X & 0 & 0 & 0 & 0 & 0 & 0 \\ X & X & X & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & X & 0 & X & 0 \\ 0 & 0 & 0 & X & X & X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & X & 0 & 0 & X & X & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X \end{bmatrix}$$

X: non-zero transition kernel elements

Figure 3-4: Structure of non-perturbed 9-state model transition kernel matrix

failed instrument, i.e. states 4, 5, 6, 7 and 8. State 9, the system loss state, is the sole element of the third class.

3.5 Decomposition of Transition Kernels into the Standard Form

At this point, it is useful the express the transition kernel in the form which comprises an eventual transition probability and a holding time density function as in Eq. (2.1) The parameters of the transition kernel elements in such form will be used for calculating the parameters of the approximate Markov process governing the class-to-class transition. The two transition kernels in Eq. (3.4) and (3.7) will be decomposed into the required form.

Eq. (3.4) can be rewritten as :

$$P_{21}(t) = \frac{\lambda_0^{\ 2}}{(\lambda_0 + 3\epsilon)^2} (\lambda_0 + 3\epsilon)^2 \, t \, e^{-(\lambda_0 + 3\epsilon)t} \tag{3.9}$$

Obviously, the second term of the RHS of the above equation is the conditional holding time PDF of transitions from states 1 to 2. The first term will be expanded in a power series in ϵ and high order terms of ϵ will be neglected, that is :

$$\begin{aligned} \frac{\lambda_0^{\ 2}}{(\lambda_0 + 3\epsilon)^2} &= 1 - \frac{6\epsilon}{\lambda_0} + o(\epsilon) \\ &\approx 1 - \frac{6\epsilon}{\lambda_0} \end{aligned} \tag{3.10}$$

Substituting in eq.(3-5), it becomes

$$P_{21}(t) = \left\{ 1 - \frac{6\epsilon}{\lambda_0} \right\} (\lambda_0 + 3\epsilon)^2 \, t \, e^{-(\lambda_0 + 3\epsilon)t} \tag{3.11}$$

The transition kernel element for transitions from state 2 to state 5 in Eq.(3.7) can be rewritten as:

$$\begin{aligned} P_{52}(t) &= \epsilon \left[\lambda_{W0} \lambda_{W1} t^2 + (\lambda_{W0} + \lambda_{W1}) t + 1 \right] e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t} \\ &= \epsilon \frac{2 \lambda_{W0} \lambda_{W1}}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3} \frac{1}{2} (\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3 \, t^2 \, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t} \\ &+ \epsilon \frac{(\lambda_{W0} + \lambda_{W1})}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^2} (\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3 \, t \, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t} \\ &+ \epsilon \frac{1}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)} (\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3 \, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t} \end{aligned} \tag{3.12}$$

It can be seen that $P_{52}(t)$ comprises 3 terms, each of which is an "eventual transition probability" times a "holding time density." Thus, the form comprises more than one term, but it will be demonstrated in Section 4.3 that those terms can be combined together to yield the standard form for the evaluation of the parameters of the approximate Markov process. A complete list of all the transition kernel elements in the standard form is included in Appendix C.

3.6 Closure

In this chapter, the structure of an example fault-tolerant system has been described. After stating the assumptions and defining all the estates, a generalized Markovian transition kernel matrix was constructed and it completely characterizes the state probability evolution. It was shown that the non-perturbed system model can be decomposed into three closed classes. Generally, any fault-tolerant system model can be decomposed into such classes if each class contains the same number of working instruments and failed instruments.

Chapter 4
Evaluation and Comparison of 9-state Model Exact and Approximate State Probability Histories

Approximate Markov process theory was developed in Chapter 2 and a fault-tolerant system model was constructed in Chapter 3. The 9-state model exact and approximate solutions will be evaluated and compared in this chapter. In the first section, the state probability histories will be calculated by a semi-Markov approach. From these results, the normalized state probability distribution that exists within each class and the total probabilities for each of the three classes will be evaluated. In the next section, the elements of the approximate technique will be deduced. That is, the stationary probability distributions of the non-perturbed process in each class and the parameters of the Markov process that approximates the behavior between the classes will be calculated. The "state" probability histories of the approximate aggregated Markov process will then be evaluated analytically. Then, the approximate state probability histories will be constructed by combining these results with the stationary probability distributions within each class. These exact and approximate results will be compared in Section 4.3.

4.1 9-state Model Numerical Results from Semi-Markov Approach

Because it is relatively easier to calculate the state probability histories numerically up to a certain number of time steps than analytically, the continuous time system representation must first be discretized into a discrete representation. The interval transition probability matrix will be calculated by using the matrix convolution sum in Eq. (1.4) and then the state probability vector at each time point is calculated by using Eq. (1.3). The initial state probability vector in Eq. (1.3) is assumed to be,

$$\underline{\pi}(0) = [\,1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]^T \tag{4.1}$$

because it is almost always the case that at the start of a mission all of the instruments are working and all of the tests are initialized.

A FORTRAN source program was written to calculate these quantities. The failure rate ϵ, of each of the instruments is assumed to be 2.5x10^{-6} sec^{-1} which is equivalent to a MTTF of 111.1 hours. The two sequential tests employed by the system are assumed to have the decision time density function parameters listed below:

$$\lambda_0 = 0.001 \qquad \lambda_{W0} = 0.05 \qquad \lambda_{F0} = 0.1$$

$$\lambda_1 = 0.05 \qquad \lambda_{W1} = 0.1 \qquad \lambda_{F1} = 0.05$$

Note that the smallest of these values (0.001) corresponds to an approximate mean time between events of 0.278 hours, which is 3 orders of magnitude shorter than the 111.1 hours MTTF of each component.

The program used double precision variables exclusively, and was run on a computer system at the Massachusetts Institute of Technology. The time step size for the discretized model was chosen as 4 seconds as a compromise between the desired mission length and the accuracy of the solution. State probability histories up to 800 time steps were calculated. This is equivalent to a mission time of 3200 seconds or just under one hour. The state probabilities at various time points between 160 seconds and 3200 seconds are shown in Figure 4-1. The evolution of the probability of occupying state 9, which is the system unreliability, is illustrated in Figure 4-2. From the state probability histories, the class probability histories can be calculated by summing the state probabilities for the states within each class. This aggregated probability histories, which will later be compared with the "state" probability histories of the approximate aggregated Markov process results, is shown in Figure 4-3 for each class. The evolution of these probabilities for the 1st class and the 2nd class is plotted in Figures 4-4 and 4-5, respectively.

The state probability distribution of the original process will be approximated by expanding the "state" probability distribution of the approximate Markov process with the stationary probability distributions of the non-perturbed process within each class, as in Eq. (1.5). Therefore, one way to measure how good the approximation is, provided the approximate Markov process gives the exact class probability distribution, is to observe how quickly and how accurately the exact normalized probability distributions[3] in class 1 and 2 approach the stationary probability distributions for these two classes. Therefore, the state probability distributions calculated above were normalized and the results are shown in Figure 4-6.

---

[3]The normalized probability distribution in a class is calculated by dividing the probability distribution elements by the total probability of occupying that class.

*** model parameters ***

ep	lam0	lam1	lamu0	lamu1	lamf0	lamf1
0.25e-05	0.10e-02	0.50e-01	0.50e-01	0.10e+00	0.10e+00	0.50e-01

*** program run parameters ***

no. of time step = 800
time step/sec. = 4.00
final time = 3200.00 sec.

normalised prob. dist. in each class

time step	state 1	2	3	4	5	6	7	8	9
40	.99276E+00	.36894E-02	.23739E-02	.39161E-03	.62376E-03	.13685E-03	.27260E-04	.18896E-04	.43271E-06
80	.98539E+00	.71651E-02	.50538E-02	.53467E-03	.14407E-02	.34827E-03	.40330E-04	.29275E-04	.17276E-05
120	.97974E+00	.96818E-02	.69896E-02	.67437E-03	.22599E-02	.56074E-03	.51955E-04	.38081E-04	.38796E-05
160	.97536E+00	.11483E-01	.83752E-02	.81376E-03	.30772E-02	.77275E-03	.63375E-04	.46667E-04	.68867E-05
200	.97189E+00	.12770E-01	.93655E-02	.95295E-03	.38928E-02	.98427E-03	.74678E-04	.55131E-04	.10747E-04
240	.96908E+00	.13688E-01	.10072E-01	.10919E-02	.47066E-02	.11953E-02	.85892E-04	.63503E-04	.15458E-04
280	.96674E+00	.14342E-01	.10575E-01	.12307E-02	.55188E-02	.14059E-02	.97038E-04	.71806E-04	.21018E-04
320	.96473E+00	.14806E-01	.10933E-01	.13691E-02	.63292E-02	.16161E-02	.10813E-03	.80054E-04	.27425E-04
360	.96297E+00	.15133E-01	.11185E-01	.15074E-02	.71381E-02	.18259E-02	.11917E-03	.88259E-04	.34679E-04
400	.96138E+00	.15363E-01	.11362E-01	.16454E-02	.79454E-02	.20352E-02	.13018E-03	.96428E-04	.42776E-04
440	.95992E+00	.15522E-01	.11485E-01	.17831E-02	.87510E-02	.22441E-02	.14115E-03	.10457E-03	.51716E-04
480	.95854E+00	.15631E-01	.11570E-01	.19206E-02	.95551E-02	.24526E-02	.15209E-03	.11268E-03	.61496E-04
520	.95724E+00	.15704E-01	.11626E-01	.20578E-02	.10358E-01	.26607E-02	.16300E-03	.12077E-03	.72115E-04
560	.95598E+00	.15751E-01	.11663E-01	.21948E-02	.11159E-01	.28684E-02	.17389E-03	.12884E-03	.83572E-04
600	.95475E+00	.15779E-01	.11685E-01	.23315E-02	.11958E-01	.30757E-02	.18475E-03	.13689E-03	.95864E-04
640	.95355E+00	.15794E-01	.11697E-01	.24679E-02	.12756E-01	.32826E-02	.19559E-03	.14492E-03	.10899E-03
680	.95237E+00	.15800E-01	.11702E-01	.26041E-02	.13552E-01	.34891E-02	.20641E-03	.15293E-03	.12295E-03
720	.95120E+00	.15798E-01	.11701E-01	.27400E-02	.14347E-01	.36952E-02	.21720E-03	.16093E-03	.13774E-03
760	.95004E+00	.15792E-01	.11697E-01	28757E-02	.15140E-01	.39008E-02	.22798E-03	.16891E-03	.15336E-03
800	.94889E+00	.15782E-01	.11690E-01	.30111E-02	.15932E-01	.41061E-02	.23873E-03	.17688E-03	.16981E-03

Figure 4-1: Exact* state probability histories of the 9-state model.
(* to within numerical round-off error)

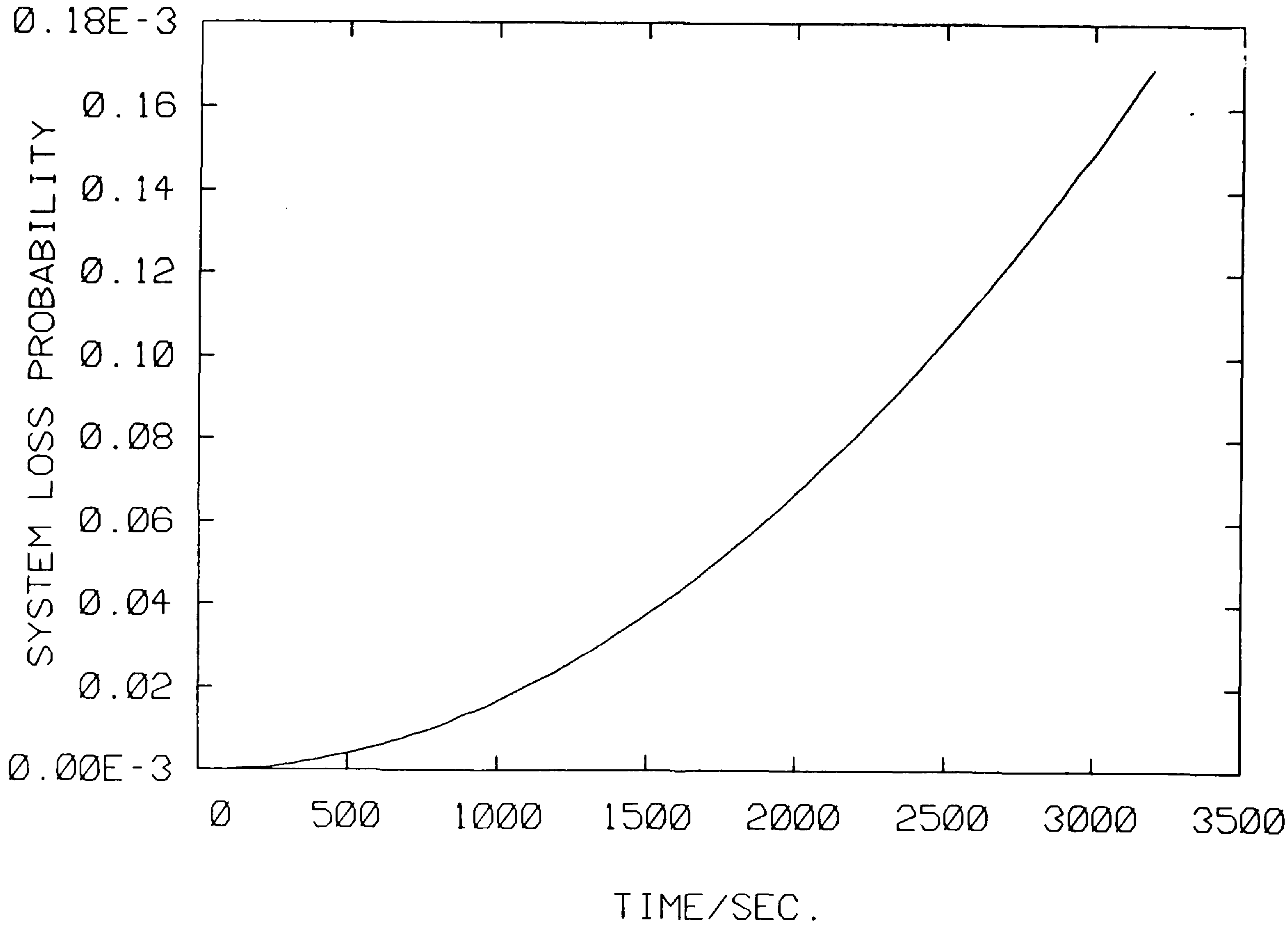

Figure 4-2: Exact system loss probability history.

*** total prob. in each class ***

time step	class1	class 2	class 3
40	0.998801194e+00	0.119837337e-02	0.432711759e-06
80	0.997604987e+00	0.239328567e-02	0.172757496e-05
120	0.996411115e+00	0.358500529e-02	0.387964282e-05
160	0.995219321e+00	0.477379211e-02	0.688667893e-05
200	0.994029419e+00	0.595983465e-02	0.107466519e-04
240	0.992841273e+00	0.714326918e-02	0.154576463e-04
280	0.991654788e+00	0.832419441e-02	0.210178290e-04
320	0.990469892e+00	0.950268212e-02	0.274254270e-04
360	0.989286537e+00	0.106787848e-01	0.346787107e-04
400	0.988104683e+00	0.118525409e-01	0.427759824e-04
440	0.986924305e+00	0.130239792e-01	0.517155682e-04
480	0.985745383e+00	0.141931211e-01	0.614958117e-04
520	0.984567902e+00	0.153599829e-01	0.721150697e-04
560	0.983391851e+00	0.165245772e-01	0.835717095e-04
600	0.982217222e+00	0.176869142e-01	0.958641062e-04
640	0.981044008e+00	0.188470018e-01	0.108990641e-03
680	0.979872203e+00	0.200048469e-01	0.122949701e-03
720	0.978701805e+00	0.211604551e-01	0.137739678e-03
760	0.977532809e+00	0.223138315e-01	0.153358966e-03
800	0.976365213e+00	0.234649808e-01	0.169805965e-03

Figure 4-3: Exact class probabilities history of 9-state model.

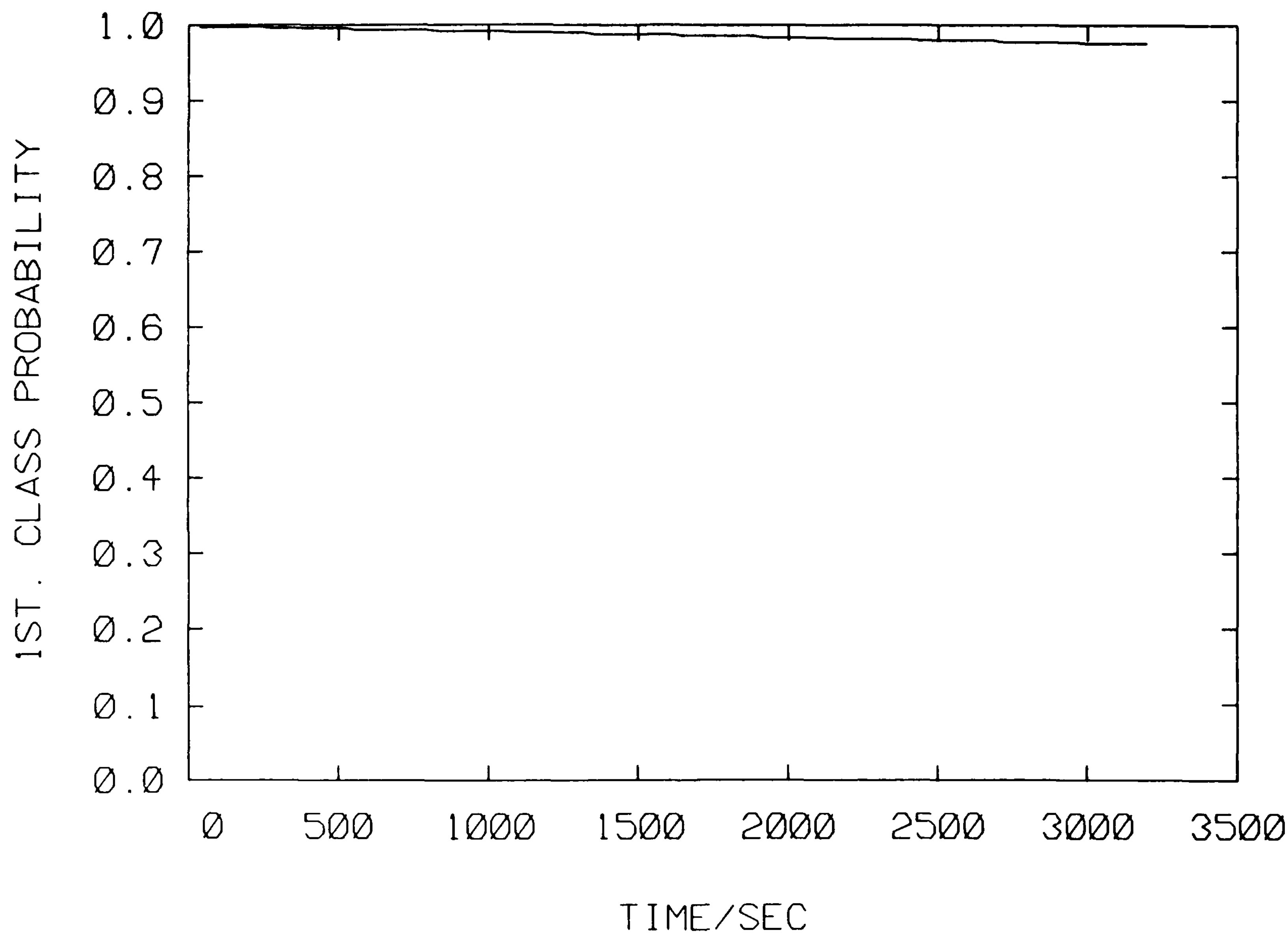

Figure 4-4: Exact class 1 probability history.

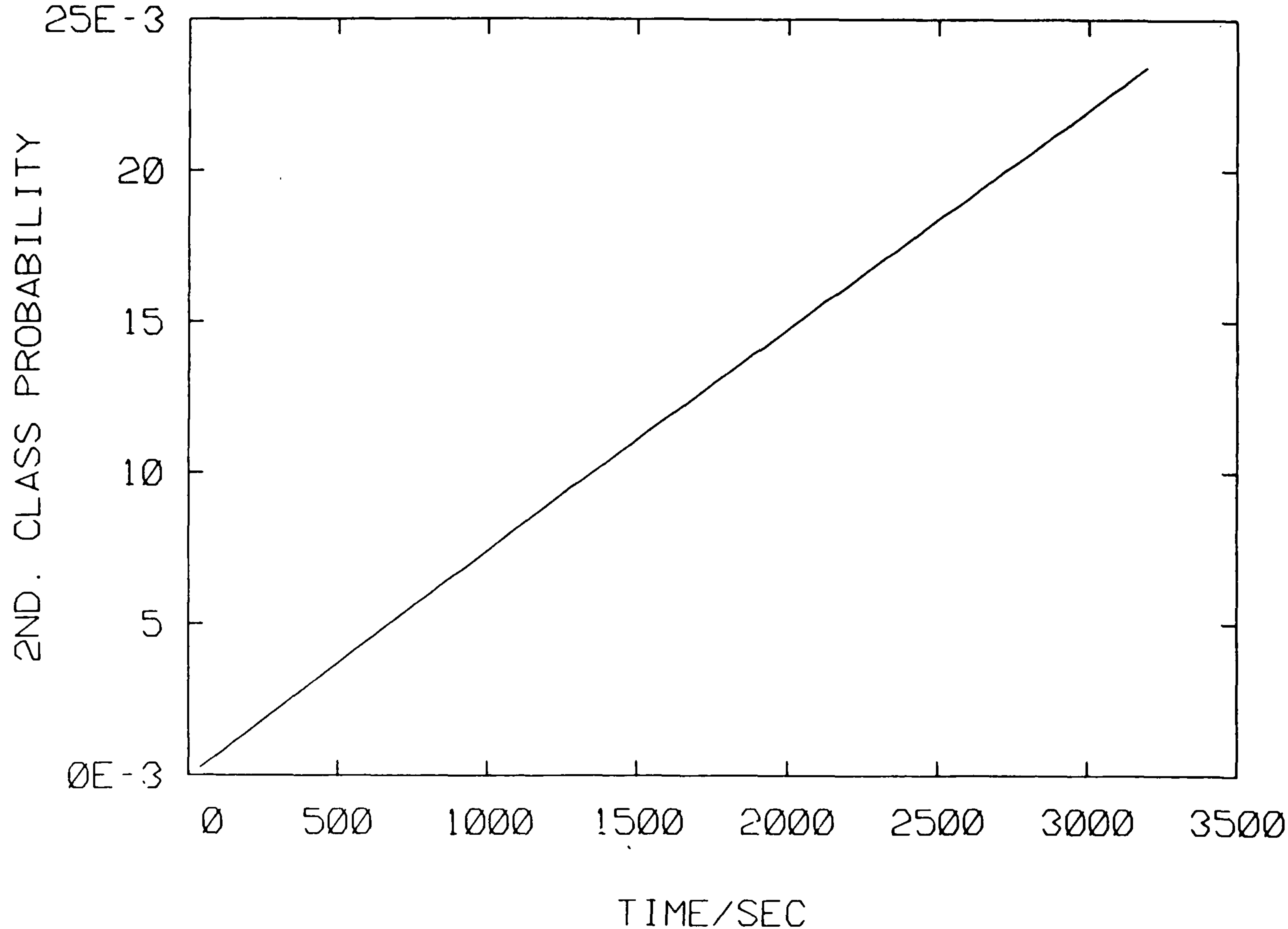

Figure 4-5: Exact class 2 probability history.

*** model parameters ***

ep	lam0	lam1	lamw0	lamw1	lamf0	lamf1
0.25e-05	0.10e-02	0.50e-01	0.50e-01	0.10e+00	0.10e+00	0.50e-01

*** program run parameters ***

no. of time step = 800
time step/sec. = 4.00
final time = 3200.00 sec.

normalised prob. dist. in each class

time step	state 1	2	3	4	5	6	7	8
40	0.993949	0.003674	0.002377	0.326783	0.520507	0.114195	0.022748	0.015768
80	0.987752	0.007182	0.005066	0.223402	0.601995	0.145520	0.016851	0.012232
120	0.983268	0.009717	0.007015	0.188107	0.630367	0.156411	0.014492	0.010622
160	0.980047	0.011538	0.008415	0.170464	0.644613	0.161873	0.013276	0.009776
200	0.977732	0.012847	0.009422	0.159895	0.653175	0.165150	0.012530	0.009250
240	0.976068	0.013787	0.010145	0.152859	0.658891	0.167336	0.012024	0.008890
280	0.974873	0.014463	0.010664	0.147840	0.662979	0.168897	0.011657	0.008626
320	0.974014	0.014948	0.011038	0.144080	0.666049	0.170068	0.011379	0.008424
360	0.973397	0.015297	0.011306	0.141158	0.668438	0.170979	0.011160	0.008265
400	0.972954	0.015548	0.011499	0.138822	0.670351	0.171709	0.010983	0.008136
440	0.972635	0.015728	0.011637	0.136912	0.671918	0.172305	0.010837	0.008029
480	0.972406	0.015857	0.011737	0.135320	0.673223	0.172802	0.010716	0.007939
520	0.972241	0.015950	0.011808	0.133974	0.674329	0.173223	0.010612	0.007863
560	0.972123	0.016017	0.011860	0.132820	0.675277	0.173583	0.010523	0.007797
600	0.972038	0.016065	0.011897	0.131821	0.676098	0.173896	0.010446	0.007740
640	0.971977	0.016099	0.011923	0.130946	0.676817	0.174169	0.010378	0.007689
680	0.971933	0.016124	0.011942	0.130174	0.677452	0.174411	0.010318	0.007645
720	0.971902	0.016142	0.011956	0.129488	0.678016	0.174625	0.010265	0.007605
760	0.971879	0.016155	0.011966	0.128875	0.678521	0.174817	0.010217	0.007570
800	0.971863	0.016164	0.011973	0.128322	0.678976	0.174990	0.010174	0.007538

Figure 4-6: Exact normalized probability distribution histories for classes 1 and 2 of the 9-state model.

4.2 Approximate State Probability Histories for the 9-State Model

4.2.1 Imbedded Markov Chains

It was shown in the last chapter that, when ϵ=0, the 9-state model decomposes into a non-perturbed model consisting of 3 closed semi-Markov chains. The eventual transition probabilities of these non-perturbed semi-Markov processes completely define the imbedded Markov chains. With the numerical values of the parameters of the model listed in Section 4.1, the transition probability matrix of the imbedded Markov chain is found and shown in Eq. (4.2).

$$P=\begin{bmatrix} 0 & 0 & 0.7407 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0.2593 & 0.2593 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.74074 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2593 & 0 & 0.7407 & 0 \\ 0 & 0 & 0 & 0.9 & 0.7407 & 0.7407 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2593 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 & 0.2593 & 0.2593 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7407 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4.2)$$

The transition probability matrix is raised to successively higher powers to characterize the behavior of the imbedded process after many transitions. It was found that when the power exceeds 40, a stationary interval transition probability matrix establishes itself as in Eq. (4.3).

By a result in Markov process theory [3], it can be concluded that the

$$
P^m = \begin{bmatrix}
0.2397 & 0.2397 & 0.2397 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.4368 & 0.4368 & 0.4368 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.3235 & 0.3235 & 0.3235 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0550 & 0.0550 & 0.0550 & 0.0550 & 0.0550 & 0 \\
0 & 0 & 0 & 0.7366 & 0.7366 & 0.7366 & 0.7366 & 0.7366 & 0 \\
0 & 0 & 0 & 0.1910 & 0.1910 & 0.1910 & 0.1910 & 0.1910 & 0 \\
0 & 0 & 0 & 0.0100 & 0.0100 & 0.0100 & 0.0100 & 0.0100 & 0 \\
0 & 0 & 0 & 0.0074 & 0.0074 & 0.0074 & 0.0074 & 0.0074 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix} \tag{4.3}
$$

decoupled imbedded Markov chains for each class are ergodic with the stationary probability vectors in each class being,

$$\underline{\pi}_M^{(1)} = [\, 0.2397 \quad 0.4386 \quad 0.3235 \,]^T \tag{4.4}$$

$$\underline{\pi}_M^{(2)} = [\, 0.0550 \quad 0.7366 \quad 0.1910 \quad 0.0100 \quad 0.0074 \,]^T \tag{4.5}$$

$$\underline{\pi}_M^{(3)} = [\, 1 \,] \tag{4.6}$$

As a result, the second condition stated in Chapter 2 is satisfied by the 9-state model and the approximate Markov process will be valid.

4.2.2 Stationary Probability Distribution of the Non-Perturbed Process

The stationary probability distribution of the non-perturbed process is needed to expand the approximate Markov process results in order to approximate the state probability distribution of the original process. By semi-Markov theory, this stationary probability distribution for each non-perturbed class is given by

$$\pi_i = \frac{\pi_{M_i} \bar{\tau}_i}{\bar{\tau}} \tag{4.7}$$

where $\bar{\tau}$ is the mean waiting time of the process:

$$\bar{\tau} = \sum_i \pi_{M_i} \bar{\tau}_i \tag{4.8}$$

and where π_{M_i} is the stationary probability for state i of the imbedded Markov process that is characterized by the eventual transition probability matrix of the semi-Markov process. $\bar{\tau}_i$ is the mean holding time in state i and it is given by,

$$\bar{\tau}_i = \sum_{all\ j} p_{ji} \bar{\tau}_{ji} \tag{4.9}$$

where p_{ji}, with the same notation before, is the eventual transition probability from state i to state j and $\bar{\tau}_{ji}$ is the mean holding time for transitions from state i to state j which is defined by,

$$\bar{\tau}_{ji} = \int_0^{\infty} t\, h_{ji}(t)\, dt \tag{4.10}$$

The calculation of the stationary probability distribution of the non-perturbed semi-Markov chain in class 1 is demonstrated here and that of class 2 is included in Appendix D.

For the non-perturbed process in class 1, state 1 can transit only to state 2, state 2 only to 2 and 3, and state 3 only to 1 and 2. The mean holding time for transitions from state i to state j is derived as follows:

Since $p_{22}(t)$ is not in the simplest form, $\bar{\tau}_{22}$ will be derived here. From the transition kernel matrix,

$$p_{22}(t) = p_{22_1} h_{22_1}(t) + p_{22_2} h_{22_2}(t)$$

where

$$p_{22_1} = \frac{2\,\lambda_{W0}^{\;2}\,\lambda_{W1}}{(\lambda_{W0} + \lambda_{W1})^3}$$

$$p_{22_2} = \frac{\lambda_{W0}^{\;2}}{(\lambda_{W0} + \lambda_{W1})^2}$$

$$h_{22_1}(t) = \tfrac{1}{2}\,(\lambda_{W0} + \lambda_{W1})^3\, t^2\, e^{-(\lambda_{W0}+\lambda_{W1})t}$$

$$h_{22_2}(t) = (\lambda_{W0} + \lambda_{W1})^2\, t\, e^{-(\lambda_{W0}+\lambda_{W1})t}$$

This can be rewritten in standard form as :

$$= p_{22}\,\{\frac{p_{22_1}}{p_{22}}\,h_{22_1}(t) + \frac{p_{22_2}}{p_{22}}\,h_{22_2}(t)\} \tag{4.11}$$

where

$$p_{22} = p_{22_1} + p_{22_2}$$

Note that any kernel element given by a sum of terms can be treated similarly. So by definition, the mean holding time $\bar{\tau}_{22}$ is given by :

$$\begin{aligned} \bar{\tau}_{22} &= \int_0^{\infty} t\,\{\,\frac{p_{22_1}}{p_{22}}\,h_{22_1}(t) + \frac{p_{22_2}}{p_{22}}\,h_{22_2}(t)\,\}\,dt \\ &= \frac{p_{22_1}}{p_{22}}\,\frac{3}{(\lambda_{W0} + \lambda_{W1})} + \frac{p_{22_2}}{p_{22}}\,\frac{2}{(\lambda_{W0} + \lambda_{W1})} \end{aligned} \tag{4.12}$$

By a similar approach,

$$\bar{\tau}_{21} = \frac{2}{\lambda_0}$$

$$\bar{\tau}_{32} = \frac{p_{32_1}}{p_{32}}\,\frac{3}{(\lambda_{W0} + \lambda_{W1})} + \frac{p_{32_2}}{p_{32}}\,\frac{2}{(\lambda_{W0} + \lambda_{W1})}$$

where

$$p_{32_1} = \frac{2\,\lambda_{w1}^{\;2}\,\lambda_{w0}}{(\lambda_{w0} + \lambda_{w1})^3}$$

$$p_{32_2} = \frac{\lambda_{w1}^{\;2}}{(\lambda_{w0} + \lambda_{w1})^2}$$

$$\bar{\tau}_{13} = \bar{\tau}_{32}$$

$$\bar{\tau}_{23} = \bar{\tau}_{22}$$

From Eq. (4.9) with the numerical values of the parameters and statistics of the 9-state model kernels substituted, the mean waiting times of the states in class 1 are,

$$\bar{\tau}_1 = p_{21}\,\bar{\tau}_{21} = 2000 \quad \textit{seconds}$$

$$\bar{\tau}_2 = p_{22}\,\bar{\tau}_{22} + p_{32}\,\bar{\tau}_{32} = 16.296 \quad \textit{seconds}$$

$$\bar{\tau}_3 = p_{13}\,\bar{\tau}_{13} + p_{23}\,\bar{\tau}_{23} = 16.296 \quad \textit{seconds}$$

With $\underline{\pi}_M$ given in Eq. (4.4), $\bar{\tau}$ is given by :

$$\bar{\tau} = \sum_i \pi_{M_i}\,\bar{\tau}_i = 491.790 \quad \textit{seconds}$$

Then the stationary probability distribution in class 1 is,

$$\pi_1 = \frac{\pi_{M_1}\,\bar{\tau}_1}{\bar{\tau}} = 0.9748$$

$$\pi_2 = \frac{\pi_{M_2}\,\bar{\tau}_2}{\bar{\tau}} = 0.0145$$

$$\pi_3 = \frac{\pi_{M_3}\,\bar{\tau}_3}{\bar{\tau}} = 0.0107$$

or,

$$\underline{\pi}^{(1)} = [\, 0.9748 \quad 0.0145 \quad 0.0107 \,]^T \tag{4.13}$$

The stationary probability distribution in class 2 is evaluated in Appendix D and the result is as follows :

$$\underline{\pi}^{(2)} = [\, 0.1250 \quad 0.6820 \quad 0.1768 \quad 0.0093 \quad 0.0069 \,]^T \tag{4.14}$$

Class 3 consists of only one state, so the stationary probability distribution is,

$$\underline{\pi}^{(3)} = [1]^T \tag{4.15}$$

4.2.3 Approximate Markov process

The Laplace transforms of the kernel elements of the approximate Markov process were derived in Chapter 2 and they involve the time scaling factor δ. But δ is only the scaling factor relating the temporal scales of the two processes. It can be set to any value and the resulting $\phi_{rk}(s)$ will be different for different values of δ. The enlarged process hence deduced will be related to the original process by the time scale factor δ set in the derivation of $\phi_{rk}(s)$.

The parameters δ, ϵ, p_{rk} and Λ_k in Eq. (2.19) for kernel element [illegible] completely define the enlarged process and p_{rk} and Λ_k can be derived [illegible] parameters of the original semi-Markov process, as shown in F[illegible] 2.20 [illegible] with the result that the enlarged process approximates the [illegible] semi-Markov process in a new time scale and any results [illegible] enlarged process must be scaled to the original time [illegible] representation of the original process is not needed [illegible] and it will not be derived here. What follows [illegible] the enlarged process of the 9-state [illegible]

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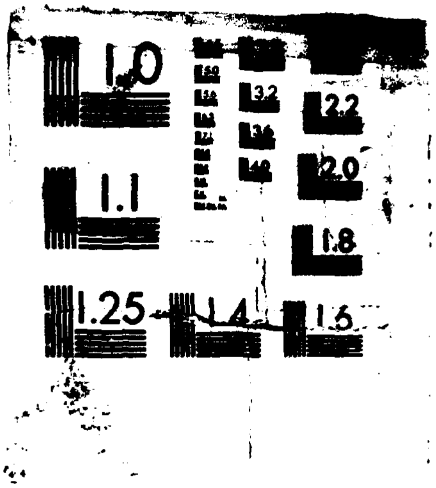

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2.5×10^{-6} sec^{-1}, the same value as the failure rate ϵ of each instrument. Then Eq. (2.19) is reduced to,

$$\phi_{rk}(s) = p_{rk} \frac{\Lambda_k}{\Lambda_k + s} \tag{4.16}$$

which is exactly the result given in [4].

The procedure for calculating Λ_1 is as follows (numerical results are quoted from Appendix C) :

$$\begin{aligned} q_1^{(1)} &= \sum_{j \in E_1} q_{j1}^{(1)} = q_{21}^{(1)} \\ &= 6000 \end{aligned} \tag{4.17}$$

$$\begin{aligned} q_2^{(1)} &= \sum_{j \in E_1} q_{j2}^{(1)} = q_{22}^{(1)} + q_{32}^{(1)} \\ &= 13.333 + 35.555 \\ &= 48.888 \end{aligned} \tag{4.18}$$

$$\begin{aligned} q_3^{(1)} &= \sum_{j \in E_1} q_{j3}^{(1)} = q_{13}^{(1)} + q_{23}^{(1)} \\ &= 35.555 + 13.333 \\ &= 48.888 \end{aligned} \tag{4.19}$$

$$\begin{aligned} a_1^{(1)} &= \sum_{j \in E_1} a_{j1} p_{j1}^{(1)} = a_{21} p_{21}^{(1)} \\ &= 2000 \end{aligned} \tag{4.20}$$

$$\begin{aligned} a_2^{(1)} &= \sum_{j \in E_1} a_{j1}\, p_{j2}^{(1)} = a_{22}\, p_{22}^{(1)} + a_{32}\, p_{32}^{(1)} \\ &= 16.296 \end{aligned} \tag{4.21}$$

$$a_3^{(1)} = \sum_{j \in E_1} a_{j3}\, p_{j3}^{(1)} = a_{13}\, p_{13}^{(1)} + a_{23}\, p_{23}^{(1)}$$
$$= 16.296 \qquad (4.22)$$

Substituting Eq. (4.17) to (4.22) into Eq. (2.22), then

$$A_1 = \frac{6000\pi_{M_1} + 48.888\pi_{M_2} + 48.888\pi_{M_3}}{2000\pi_{M_1} + 16.296 g(p)_{M_2} + 16.296\pi_{M_3}}$$
$$= 3 \qquad (4.23)$$

The procedure for obtaining p_{21} is as follows :

$$q_1^{(21)} = \sum_{j \in E_2} q_{j1}^{(1)} = q_{41}^{(1)}$$
$$= 6000 \qquad (4.24)$$

$$q_2^{(21)} = \sum_{j \in E_2} q_{j2}^{(1)} = q_{52}^{(1)} + q_{72}^{(1)}$$
$$= 48.888 \qquad (4.25)$$

$$q_3^{(21)} = \sum_{j \in E_2} q_{j3}^{(1)} = q_{63}^{(1)} + q_{83}^{(1)}$$
$$= 48.888 \qquad (4.26)$$

Substituting Eq. (4.20) to (4.22) and Eq. (4.24) to (4.26) into Eq. (2.21), then

$$p_{21} = \frac{6000\pi_{M_1} + 48.888\pi_{M_2} + 48.888\pi_{M_3}}{2000\pi_{M_1} + 16.296\pi_{M_2} + 16.296\pi_{M_3}}$$
$$= 1 \qquad (4.27)$$

p_{21} equal to 1 implies that p_{31} equals 0 due to the fact that the sum of the eventual transition probabilities exiting a state must be 1.

The procedure for obtaining A_2 is as follows :

$$q_4^{(2)} = \sum_{j \in E_2} q_{j4}^{(2)} = q_{54}^{(2)} + q_{74}^{(2)}$$
$$= 80 \tag{4.28}$$

$$q_5^{(2)} = \sum_{j \in E_2} q_{j5}^{(2)} = q_{55}^{(2)} + q_{65}^{(2)}$$
$$= 32.593 \tag{4.29}$$

$$q_6^{(2)} = \sum_{j \in E_2} q_{j6}^{(2)} = q_{46}^{(2)} + q_{56}^{(2)}$$
$$= 32.593 \tag{4.30}$$

$$q_7^{(2)} = \sum_{j \in E_2} q_{j7}^{(2)} = q_{77}^{(2)} + q_{87}^{(2)}$$
$$= 32.593 \tag{4.31}$$

$$q_8^{(2)} = \sum_{j \in E_2} q_{j8}^{(2)} = q_{48}^{(2)} + q_{78}^{(2)}$$
$$= 32.593 \tag{4.32}$$

$$a_4^{(2)} = \sum_{j \in E_2} a_{j4}\, p_{j4}^{(2)} = a_{54}\, p_{54}^{(2)} + a_{74}\, p_{74}^{(2)}$$
$$= 40 \tag{4.33}$$

$$a_5^{(2)} = \sum_{j \in E_2} a_{j5}\, p_{j5}^{(2)} = a_{55}\, p_{55}^{(2)} + a_{65}\, p_{65}^{(2)}$$
$$= 16.296 \tag{4.34}$$

$$a_6^{(2)} = \sum_{j \in E_2} a_{j6}\, p_{j6}^{(2)} = a_{46}\, p_{46}^{(2)} + a_{56}\, p_{56}^{(2)}$$
$$= 16.296 \tag{4.35}$$

$$a_7^{(2)} = \sum_{j \in E_2} a_{j7}\, p_{j7}^{(2)} = a_{77}\, p_{77}^{(2)} + a_{87}\, p_{87}^{(2)}$$
$$= 16.296 \tag{4.36}$$

$$a_8^{(2)} = \sum_{j \in E_2} a_{j8}\, p_{j8}^{(2)} = a_{48}\, p_{48}^{(2)} + a_{78}\, p_{78}^{(2)}$$
$$= 16.296 \tag{4.37}$$

substituting Eq. (4.28) to (4.37) into Eq. (4.22), then

$$A_2 = \frac{80\, \pi_{M_4} + 32.593\, \pi_{M_5} + 32.593\, \pi_{M_6} + 32.593\, \pi_{M_7} + 32.593\, \pi_{M_8}}{40\, \pi_{M_4} + 16.292\, \pi_{M_5} + 16.292\, \pi_{M_6} + 16.292\, \pi_{M_7} + 16.292\, \pi_{M_8}}$$
$$= 2 \tag{4.38}$$

The procedure for obtaining p_{32} :

$$q_4^{(32)} = \sum_{j \in E_3} q_{j4}^{(2)} = q_{94}^{(2)}$$
$$= 80 \tag{4.39}$$

$$q_5^{(32)} = \sum_{j \in E_3} q_{j5}^{(2)} = q_{95}^{(2)}$$
$$= 32.593 \tag{4.40}$$

$$q_6^{(32)} = \sum_{j \in E_3} q_{j6}^{(2)} = q_{96}^{(2)}$$
$$= 32.593 \tag{4.41}$$

$$q_7^{(32)} = \sum_{j \in E_3} q_{j7}^{(2)} = q_{97}^{(2)}$$
$$= 32.593 \qquad (4.42)$$

$$q_8^{(32)} = \sum_{j \in E_3} q_{j8}^{(2)} = q_{98}^{(2)}$$
$$= 32.593 \qquad (4.43)$$

substituting Eq. (4.28) to (4.32) and Eq. (4.39) to (4.43) into Eq. (2.21), then

$$p_{32} = \frac{80\ \pi_{M_4} + 32.593\ \pi_{M_5} + 32.593\ \pi_{M_6} + 32.593\ \pi_{M_7} + 32.593\ \pi_{M_8}}{80\ \pi_{M_4} + 32.593\ \pi_{M_5} + 32.593\ \pi_{M_6} + 32.593\ \pi_{M_7} + 32.593\ \pi_{M_8}}$$
$$= 1 \qquad (4.44)$$

p_{32} equal to 1 implies that p_{12} equals zero.

Since class 3 is a trapping class, it will not affect the result if it is assumed that the numerical values of A_3 and p_{33} are both 1. By summing all the results obtained above, the Laplace transform of the transition kernel matrix of the approximate aggregated Markov process is as follows :

$$P^a(s) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{3}{s+3} & 0 & 0 \\ 0 & \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix} \qquad (4.45)$$

or in the scaled time domain,

$$P^a(t) = \begin{bmatrix} 0 & 0 & 0 \\ 3e^{-3t'} & 0 & 0 \\ 0 & 2e^{-2t'} & e^{-t'} \end{bmatrix} \qquad (4.46)$$

By semi-Markov theory, the Laplace transform of the interval transition matrix can be expressed as follows :

$$\Phi(s) = [\, sI + (I - P)\Lambda \,]^{-1} \tag{4.47}$$

where s is the Laplace operator, I is the identity matrix, P is the eventual transition probability matrix and Λ is a diagonal matrix whose i-th element is the exponential transition rate out of "state" i. So, for the approximate Markov process for the 9-state model, P and Λ are as follows :

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{4.48}$$

$$\Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.49}$$

Substituting Eq. (4.48) and (4.49) into Eq. (4.47) yields, after some manipulations,

$$\Phi^e(s) = \begin{bmatrix} \frac{1}{s+3} & 0 & 0 \\ \frac{3}{(s+2)(s+3)} & \frac{1}{(s+2)} & 0 \\ \frac{6}{s(s+2)(s+3)} & \frac{2}{s(s+2)} & \frac{1}{s} \end{bmatrix} \tag{4.50}$$

or in the scaled time domain,

$$\Phi^{e}(t) = \begin{bmatrix} e^{-3t'} & 0 & 0 \\ 3(e^{-2t'} - e^{-3t'}) & e^{-2t'} & 0 \\ 1 - 3e^{-2t'} + 2e^{-3t'} & 1 - 2e^{-2t'} & 1 \end{bmatrix} \tag{4.51}$$

Since the initial state probability vector used in the exact state probability distribution histories calculation was assumed to be $\underline{\pi}(0) = [\,1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\,]^T$, the state probability vector for the approximate aggregated Markov process will be,

$$\underline{\pi}^{e}(0) = [\,1 \quad 0 \quad 0\,]^T \tag{4.52}$$

By Eq. (1.1), the "state" probabilities of the approximate aggregated Markov process are,

$$\pi_1^e(t') = e^{-3t'} \tag{4.53}$$

$$\pi_2^e(t') = 3(e^{-2t'} - e^{-3t'}) \tag{4.54}$$

$$\pi_2^e(t') = 1 - 3e^{-2t'} + 2e^{-3t'} \tag{4.55}$$

The argument t'is used here in order to distinguish the different time scale used for the approximate Markov process. The original semi-Markov 9-state model is defined in a faster time scale, denoted t. If the $\pi_i^e(t')$ are expressed in this original temporal scale, then Eq. (4.53) to (4.55) will, in general, become :

$$\pi_1^e(t) = e^{-3\delta t} \tag{4.56}$$

$$\pi_2^e(t) = 3(e^{-2\delta t} - e^{-3\delta t}) \tag{4.57}$$

$$\pi_3^e(t) = 1 - 3e^{-2\delta t} + 2e^{-3\delta t} \tag{4.58}$$

or, in this example where $\delta = \epsilon$, these three equations become,

$$\pi_1^e(t) = e^{-3\epsilon t} \tag{4.59}$$

$$\pi_2^e(t) = 3(e^{-2\epsilon t} - e^{-3\epsilon t}) \tag{4.60}$$

$$\pi_3^e(t) = 1 - 3e^{-2\epsilon t} + 2e^{-3\epsilon t} \tag{4.61}$$

By expanding the approximate Markov process with the stationary probability distributions of the non-perturbed decoupled processes obtained in Eq. (4.13) to Eq. (4.15), the approximate probability distribution for the 9-state model is,

$$\underline{\pi}^a(t) = \begin{array}{l} \begin{bmatrix} 0.9748 \\ 0.0145 \\ 0.0107 \end{bmatrix} e^{-3\epsilon t} \\ \begin{bmatrix} 0.1250 \\ 0.6820 \\ 0.1768 \\ 0.0093 \\ 0.0069 \end{bmatrix} 3(e^{-2\epsilon t} - e^{-3\epsilon t}) \\ \begin{bmatrix} 1 \end{bmatrix} (1 - 3e^{-2\epsilon t} + 2e^{-3\epsilon t}) \end{array} \tag{4.62}$$

4.3 Comparison and Discussion of Results

The accuracy of the approximate approach depends on two key factors. The first factor is how quickly and how accurately the normalized probability distribution in each class of the 9-state model converges to the non-perturbed stationary probability distribution. The second factor is how accurate the "state" probabilities of the approximate aggregated Markov process are relative to the class probabilities of the 9-state model. The comparison of results for the example system in these two aspects follows:

First, the normalized probability distributions at the end of the mission, i.e. at t = 3200 sec., obtained in Figure 4-1 and the analytical non-perturbed stationary probability distributions obtained in Section 4.2.2 are compared in Figure 4-7. The largest and the smallest relative percentage errors occur in state 3 and 5, respectively. The normalized probability trajectories for states 3 and 5 are plotted in Figures 4-8 and 4-9 along with the corresponding analytical stationary probability distribution value (a constant in each case).

In Figure 4-8 the state probability trajectory in state 3 starts to converge to within 12% of the stationary value from t = 800 sec. onward and at the end of the mission it converges to a value of 0.012, which is higher than the stationary probability. In Figure 4-9, the state 5 normalized probability trajectory converges faster to within 10% of the stationary probability from t = 350 sec. onward and converges to within 0.5% or to a value of 0.68 at the end of the mission. The main contribution to the large percentage error of the normalized probabilities in states 2 and 3 relative to the analytical stationary probabilities is due to the large step size chosen for the discretization of the 9-state model (step size was chosen for compromise between accuracy and mission length). To illustrate this point, the

class	state	normalized probability distribution	stationary probability distribution (numerical)	stationary probability distribution (analytical)	relative % error
1	1	0.9718	0.9719	0.9748	-3.0
	2	0.0161	0.0162	0.0145	11.7
	3	0.0119	0.0120	0.0107	11.7
2	4	0.1283	0.1179	0.1250	2.7
	5	0.6789	0.6876	0.6820	-0.5
	6	0.1749	0.1783	0.1768	-1.0
	7	0.0101	0.0094	0.0093	9.6
	8	0.0075	0.0069	0.0069	9.7

Figure 4-7: Comparison of normalized probability distribution at t=3200 sec. and *stationary probability distribution of the non-perturbed process*

stationary state probability distribution within each class was obtained numerically by running the 9-state model program for 800 time steps with $\epsilon=0$ and $\pi(0) = [\,1\;0\;0\;1\;0\;0\;0\;0\;1\,]$. The result is the numerical stationary probability distribution of the non-perturbed process which is shown in Figure 4-7. The class 1 normalized probability distribution converges to the numerical stationary probability distribution rather than the analytical stationary probability distribution. From this, it can be concluded that the base-line results for the normalized probability distribution are in error due to computational effects in the

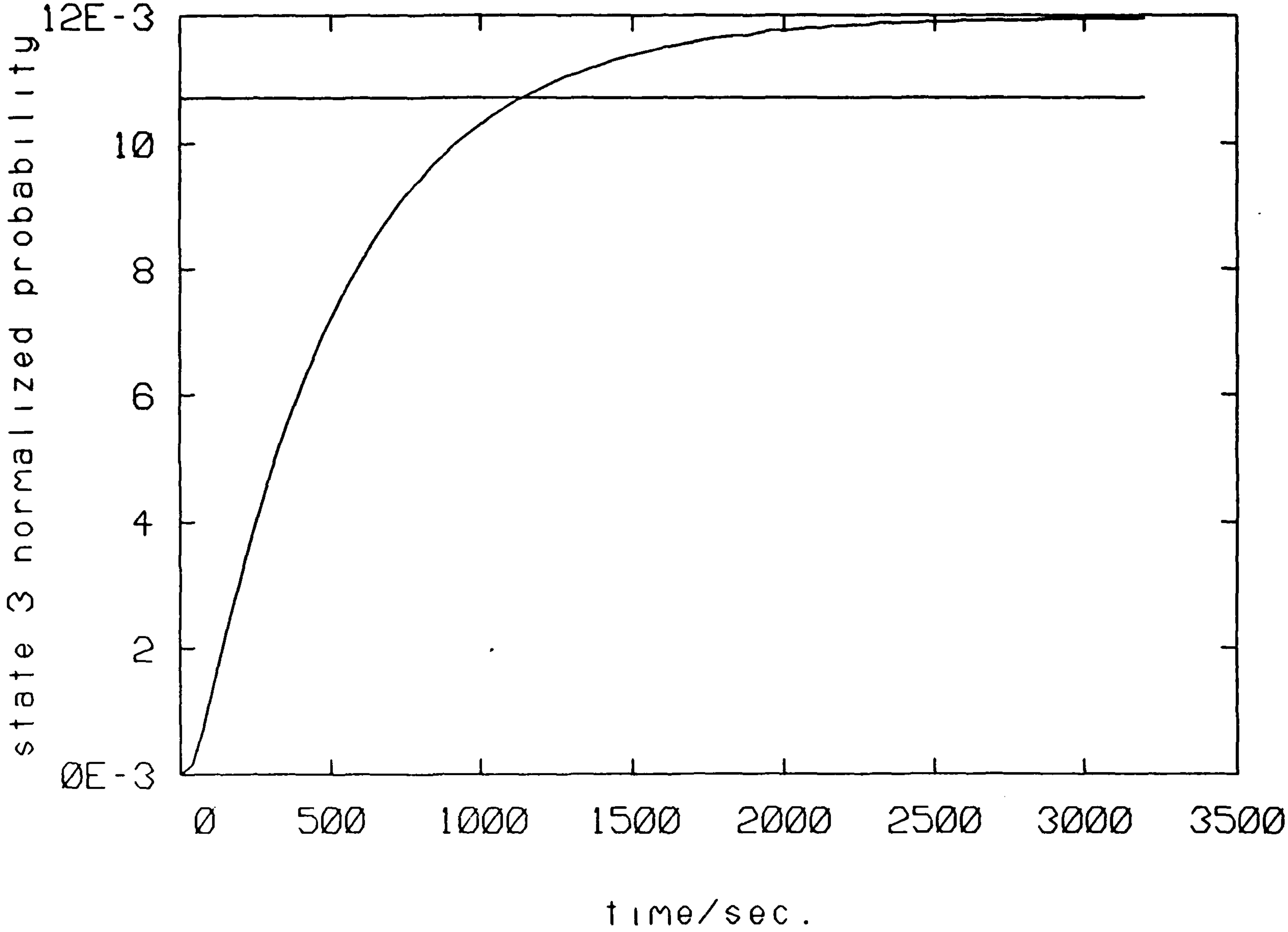


Figure 4-8: State 3 normalized probability history.

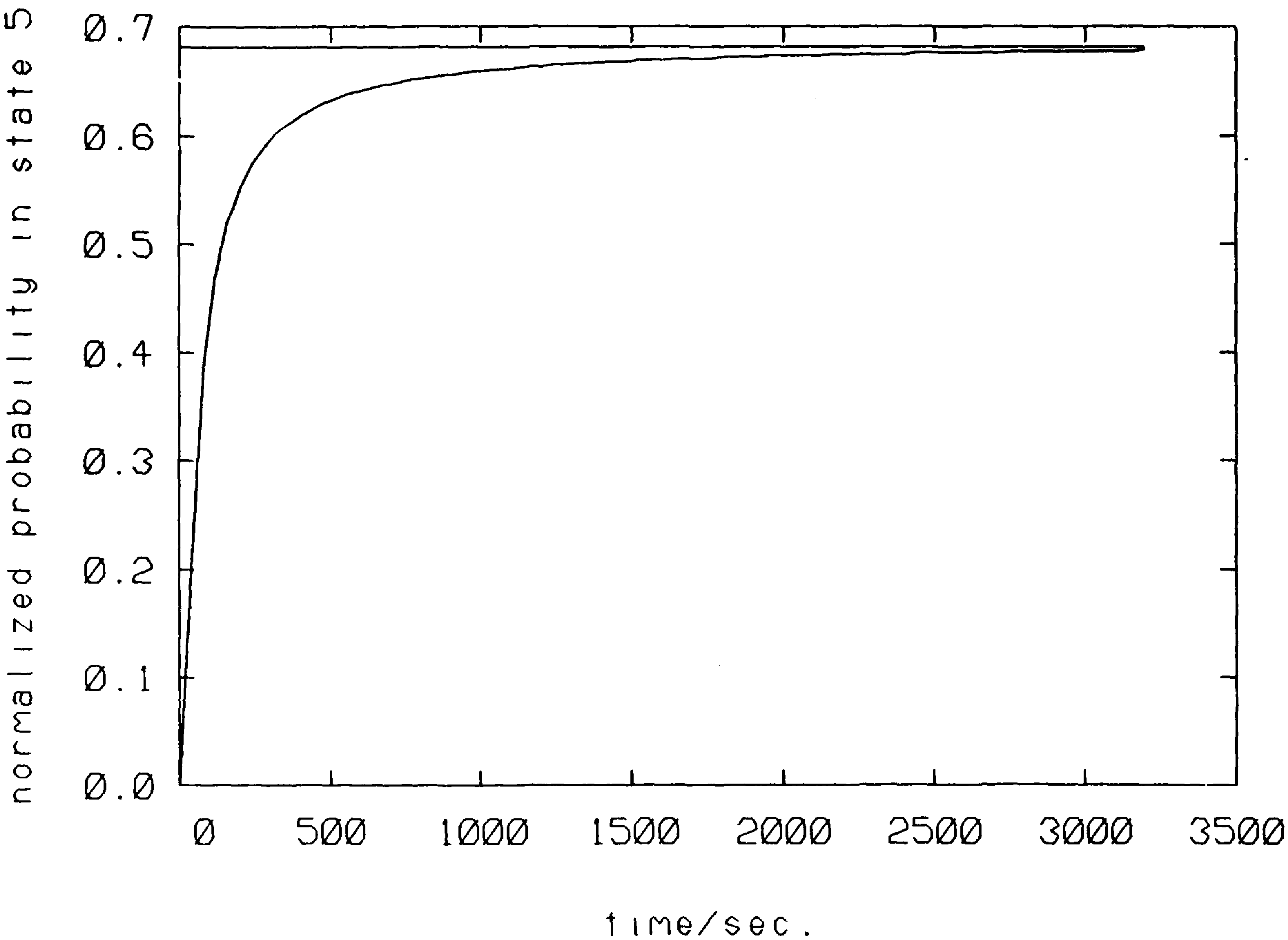


Figure 4-9: State 5 normalized probability history.

discretization of the governing matrix of the model.

The class probability trajectories from the semi-Markov approach were obtained in Section 4.1 and the "state" probabilities of the approximate aggregated Markov process in closed form were deduced in Section 4.2.3. The results of these two different approaches at 40, 80, and 800 time steps are compared in Figure 4-10.

t/sec.	class	numerical semi-Markov approach	approximate Markov process technique
160	1	0.9988	0.9988
	2	0.1198e-2	0.1199e-2
	3	0.4327e-6	0.4804e-6
320	1	0.9976	0.9976
	2	0.2393e-2	0.2395e-2
	3	0.1728e-5	0.1918e-2
3200	1	0.9764	0.9763
	2	0.2346e-1	0.2352e-1
	3	0.1698e-3	0.1895e-3

Figure 4-10: Comparison of class probability obtained from numerical semi-Markov approach and from approximate Markov process technique

Figure 4-10 indicates that the largest absolute error between the two results is only

0.0001 in class 1 at the end of the mission. This shows that the enlarged process approximates the aggregated probability distribution of the exact model very well. With the inaccurate base-line results taken into account, the absolute error of the approximate state probability distribution, obtained by expanding the approximate Markov process with the analytical stationary probability distribution as in Eq. (4.62), will be less than 0.0000117 for any state beyond $t = 800$ sec. This is only 1/2000 of the MTTF of each instrument.

The high accuracy of the enlarged process approximation led to a closer examination of the example system. Undoubtedly, the model for this system is a "pure" semi-Markov model because none of the transition kernel elements has an exponentially distributed holding time. If the states in each class are examined carefully, it can be found that all the states within each class represent the same number of working and failed instruments. By combinatorial analysis, the time to transition from class 1 to class 2 is exponentially distributed with a parameter of 3ϵ. So the 9-state model class to class transition is intrinsically governed by a Markov process. Although the 9-state model has this property, it is not non-trivial to deduce whether the approximate technique did accurately approximate the state probability distribution of a genuine semi-Markov process.

Because of this special property of the 9-state model, the approximate technique will be further tested on several semi-Markov models in the next chapter.

Chapter 5
Further Tests of Approximate Technique with 4-State Models

The approximate technique applied to the 9-state model was demonstrated in the previous chapter. In order to further test the technique for other models that a fault-tolerant system might produce without expending a lot of effort to create large state space models, several relatively small 4-state semi-Markov process models will be created in this chapter to simulate various fault-tolerant system class to class transition structures and properties, and to evaluate the results of the approximation technique.

There are five models to be examined in this chapter. Their detailed descriptions appear below, but they will be summarized here. In case I, there are two ergodic classes where the second class is a trapping class. Case II has the property that ergodic class 1 can transit to trapping classes 3 or 4. The difference between this case and Case III is that in Case III, class 2 can transit back to class 1. The next example, Case IV, consists of two non-ergodic classes where class 2 is a trapping class. Case V, the last model, comprises four classes, where classes 3 and 4 can be entered from both class 1 and class 2.

5.1 Case I

In this model, the semi-Markov process consists of four states, The state transition diagram is shown in Figure 5-1.

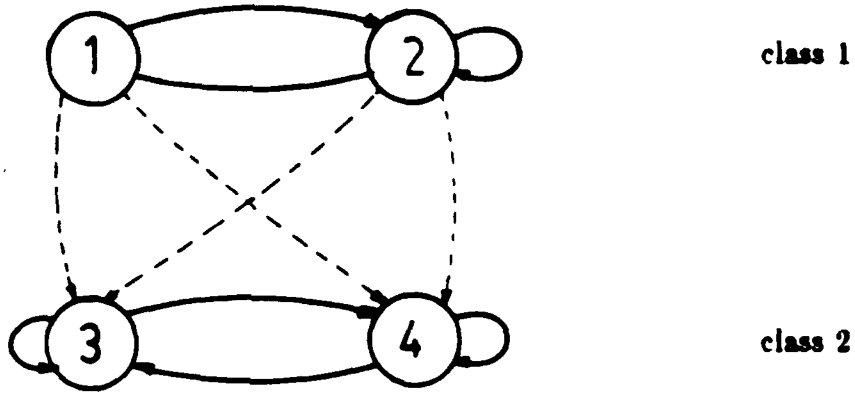


Figure 5-1: State transition diagram for Case I

The process can be decomposed into two classes, namely class 1 and class 2, when $\epsilon=0$. Class 1 comprises states 1 and 2 and class 2 comprises states 3 and 4. The transition from class 1 to class 2 is through the small eventual transition probability in terms of ϵ from states 1 and 2 to states 3 and 4. However state 3 and state 4 cannot transit back to any of the states in class 1, hence class 2 is a trapping class. The governing transition kernel matrix is given by the following:

$$P(t)=\begin{bmatrix} 0 & (0.3-7\epsilon)\lambda_1 e^{-\lambda_1 t} & 0 & 0 \\ (1-6\epsilon)\lambda_2 e^{-\lambda_2 t} & (0.7-2\epsilon)\lambda_2 e^{-\lambda_2 t} & 0 & 0 \\ 2\epsilon\lambda_1^2 te^{-\lambda_1 t} & 6\epsilon\lambda_1^2 te^{-\lambda_1 t} & 0.4\lambda_1^2 te^{-\lambda_1 t} & 0.5\lambda_1^2 te^{-\lambda_1 t} \\ 4\epsilon\lambda_2^2 te^{-\lambda_2 t} & 3\epsilon\lambda_2^2 te^{-\lambda_2 t} & 0.6\lambda_2^2 te^{-\lambda_2 t} & 0.5\lambda_2^2 te^{-\lambda_2 t} \end{bmatrix} \quad (5.1)$$

where $\lambda_1=0.2$, $\lambda_2=0.1$, $\epsilon=2.5\text{x}10^{-6}$, (all units are in $\sec^{-1}$).
It is assumed the initial condition is,

$$\pi(0)=[1 \quad 0 \quad 0 \quad 0]^T \quad (5.2)$$

One point about this model to be emphasized is that the holding time density functions for the transitions from states in class 1 to states in class 2 and those within class 2 are 2nd order Erlang PDFs. These are non-exponential holding time density functions, so the model is a semi-Markov process.

Stationary probability distribution of the non-perturbed semi-Markov process

By setting $\epsilon=0$ and dropping all the holding time density functions in the transition kernel matrix, the transition probability matrix of the non-perturbed Markov process is found to be :

$$P=\begin{bmatrix} 0 & 0.3 & 0 & 0 \\ 1 & 0.7 & 0 & 0 \\ 0 & 0 & 0.4 & 0.5 \\ 0 & 0 & 0.6 & 0.5 \end{bmatrix} \quad (5.3)$$

By raising the single step transition probability matrix successively to higher

powers, the stationary interval transition probability matrix is found to be :

$$P^{\infty}=\begin{bmatrix} 0.2308 & 0.2308 & 0 & 0 \\ 0.7692 & 0.7692 & 0 & 0 \\ 0 & 0 & 0.4545 & 0.4545 \\ 0 & 0 & 0.5455 & 0.5455 \end{bmatrix} \tag{5.4}$$

Then the stationary probability vectors of the non-perturbed imbedded Markov processes in class 1 and 2 are:

$$\underline{\pi}_M^{(1)}=[\,0.2308 \quad 0.7692\,]^T \tag{5.5a}$$

$$\underline{\pi}_M^{(2)}=[\,0.4545 \quad 0.5455\,]^T \tag{5.5b}$$

The mean waiting times for the states in class 1 are,

$$\tau_1=p_{21}\frac{1}{\lambda_2}=10 \tag{5.6}$$

$$\tau_2=p_{12}\,\frac{1}{\lambda_1}+p_{22}\frac{1}{\lambda_2}=8.5 \tag{5.7}$$

Therefore the mean waiting time of the process in class 1 is

$$\tau=\sum_{i\in E_1}\pi_{M_i}\tau_i=8.8462 \tag{5.8}$$

Hence, the stationary probabilities in class 1 are

$$\pi_1^{(1)}=\frac{\pi_{M_1}\tau_1}{\tau}=0.2609 \tag{5.9a}$$

$$\pi_2^{(1)}=\frac{\pi_{M_1}\tau_1}{\tau}=0.7391 \tag{5.9b}$$

or in vector form,

$$\underline{\pi}^{(1)} = [\, 0.2609 \quad 0.7391 \,]^T \tag{5.10}$$

The mean waiting times for the states in class 2 are:

$$\bar{\tau}_3 = p_{33}\frac{2}{\lambda_1} + p_{43}\frac{2}{\lambda_2} = 16 \tag{5.11}$$

$$\bar{\tau}_4 = p_{34}\frac{2}{\lambda_1} + p_{44}\frac{2}{\lambda_2} = 15 \tag{5.12}$$

Therefore the meaning waiting time of the process in class 2 is,

$$\bar{\tau} = \sum_{i \in E_2} \pi_{M_i} \bar{\tau}_i = 15.4545 \tag{5.13}$$

Hence, the stationary probabilities in class 2 are,

$$\pi_3^{(2)} = \frac{\pi_{M_3} \bar{\tau}_3}{\bar{\tau}} = 0.4705 \tag{5.14a}$$

$$\pi_4^{(2)} = \frac{\pi_{M_4} \bar{\tau}_4}{\bar{\tau}} = 0.5295 \tag{5.14b}$$

or in vector form,

$$\underline{\pi}^{(2)} = [\, 0.4705 \quad 0.5295 \,]^T \tag{5.15}$$

Approximate Markov process

In all the five cases in this chapter, the time scale factor δ is set equal to ϵ and a similar approach as in the 9-state model example in the last chapter is used for evaluating the approximate Markov process.

The Laplace transform of the kernel element for transition from aggregated "state" 1 to aggregated "state" 2 is given by:

$$\phi_{21}(s) = p_{21} \frac{A_1}{s + A_1} \tag{5.16}$$

where

$$A_1 = \frac{\sum_{i \in E_1} \pi_{M_i}^{(1)} \varphi_i^{(1)}}{\sum_{i \in E_1} \pi_{M_i}^{(1)} r_i^{(1)}} \tag{5.17}$$

From the transition kernel matrix in eq.(5.1),

$$\varphi_1^{(1)} = 6 \tag{5.18}$$

$$\varphi_2^{(1)} = 7 + 2 = 9 \tag{5.19}$$

Substituting all the numerical quantities in eq.(5.15):

$$A_1 = \frac{0.2308 \times 6 + 0.7692 \times 9}{0.2308 \times 10 + 0.7692 \times 8.5} = 0.9391 \tag{5.20}$$

Obviously from the structure of the class to class transitions,

$$p_{21} = 1 \tag{5.21}$$

Therefore,

$$\phi_{21}(s) = \frac{0.9391}{s + 0.9391} \tag{5.22}$$

or in the scaled time domain,

$$\phi_{21}(t') = 0.9391\, e^{-0.9391\, t'} \tag{5.23}$$

Since there are only two classes and class 2 cannot transit to class 1, the approximate probability in class 2 is given by,

$$\pi_2^e(t') = \int_0^{t'} \phi_{21}(\tau)\, d\tau$$
$$= 1 - e^{-0.9391\, t'} \tag{5.24}$$

and the approximate probability in class 1 is,

$$\pi_1^e(t') = 1 - \pi_2^e(t')$$
$$= e^{-0.9391\, t'} \tag{5.25}$$

In the original time scale, this becomes

$$\pi_1^e(t) = e^{-0.9391 \times 2.5 \times 10^{-6}\, t} \tag{5.26}$$

$$\pi_2^e(t) = 1 - e^{0.9391 \times 2.5 \times 10^{-6}\, t} \tag{5.27}$$

Exact solution of the original semi-Markov process

The exact solution of the semi-Markov process is to be evaluated analytically by using eq.(4.40). Although there are only four states in the model, the manipulation will have to be helped by using a powerful symbolic manipulation program called MACSYMA which resides in the Multics system at the Massachusetts Institute of Technology. Two of the elements of the interval transition probability matrix are obtained as follows:

$$\begin{aligned}\Phi_{11}(t) = {} & 0.030115\,[\, e^{-2.35e-6\, t} - e^{-0.22999815\, t}\,] \\ & + 0.230749\,[\, e^{-2.35e-6\, t} - e^{-0.22999815\, t}\,] \\ & + 5.384780e{-7}\; t\, e^{-0.1\, t} + 0.538474\, e^{-0.1\, t} \\ & + 2.833533e{-5} e^{-0.2\, t}\end{aligned} \tag{5.28}$$

$$\Phi_{21}(t) = 0.939775\,[\,e^{-2.35\,e-6\,t} - e^{0.22999815\,t}\,]$$
$$+\,0.538533\,[\,e^{-2.35\,e-6\,t} - e^{0.22999815\,t}\,]$$
$$-(\,5.769362e-7t + 0.538483\,)e^{-0.1\,t} - 5.000e-5\,e^{-0.2\,t} \tag{5.29}$$

Since the initial condition was assumed in Eq. (5.2) to be $\underline{\pi}(0) = [\,1 \quad 0 \quad 0 \quad 0\,]^T$, then

$$\pi_1(t) = \Phi_{11}(t) \tag{5.30}$$

$$\pi_2(t) = \Phi_{21}(t) \tag{5.31}$$

Therefore the total probability in class 1 is,

$$P_{E_1}(t) = \pi_1(t) + \pi_2(t) \tag{5.32}$$

Comparison of results

The approximate and exact total probabilities in class 1 at different time points are compared in Figure 5-2. The results indicate that errors in the aproximation occur only at the fifth decimal places through the time history up to t=10000 with the maximum relative percentage error occuring at t=1000 at value of only 0.0002%. This shows that the class probability is well approximated by the enlarged process.

After the class probability results have been compared, the normalized probability distribution within class 1 is compared with the stationary probability distribution that is given in Eq. (5.9). The normalized probability distribution history in class 1 is shown in Figure 5-3.

By comparing the stationary normalized probability distribution that was established after 100 seconds with the stationary probability distribution of the non-perturbed semi-Markov process in class 1 obtained in Eq. (5.9), it is found that

t/sec.	approximate class probability $\pi_1^e(t)$	exact class probability $P_{E_1}(t)$
1	0.99999	1.00000
5	0.99999	1.00000
10	0.99998	0.99999
50	0.99988	0.99990
100	0.99977	0.99978
500	0.99883	0.99884
1000	0.99765	0.99767
5000	0.98833	0.98834
10000	0.97680	0.97679

Figure 5-2: Comparison of approximate and exact probability in class 1

there is no error up to 4 decimal places. This implies that the semi-Markov process is well approximated to within 0.0002% error after the transient period of 100 seconds at the beginning of the mission. The transient period is about 10 times the maximum mean waiting time among the states of the non-perturbed process in class 1 and 0.025% of the MTTF.

t/sec.	state 1	state 2
1	0.9075	0.0925
5	0.6510	0.3490
10	0.4791	0.5209
40	0.2707	0.7293
100	0.2609	0.7391
200	0.2609	0.7391
600	0.2609	0.7391

Figure 5-3: Normalized probability distribution in class 1

5.2 Case II

In Case I, the semi-Markov process was well-approximated by the enlarged process after the transient period. However, the model there is not general enough to include different classes that class 1 can transit to, as is likely to be the case for many fault-tolerant system models. Ironically, in the 9-state model there is not a class that can transit to both of the other two classes. In order to investigate how valid the approximation is, another model will be formed. It consists of four states which decompose into three classes. Class 1 comprises states 1 and 2; classes 2 and 3 are just states 3 and 4, respectively. Class 1 can transit to classes 2 and 3 while classes 3 and 4 are trapping classes. The state transition diagram is shown in Figure 5-4 and the process' governing transition kernel matrix is defined as follows:

$$P(t)=\begin{bmatrix} 0 & (0.3-7\epsilon)\lambda_1 e^{-\lambda_1 t} & 0 & 0 \\ (1-8\epsilon)\lambda_2 e^{-\lambda_2 t} & (0.7-2\epsilon)\lambda_2 e^{-\lambda_2 t} & 0 & 0 \\ 2\epsilon\lambda_3 e^{-\lambda_3 t} & 6\epsilon\lambda_3 e^{-\lambda_3 t} & \lambda_3 e^{-\lambda_3 t} & 0 \\ 4\epsilon\lambda_4 e^{-\lambda_4 t} & 3\epsilon\lambda_4 e^{-\lambda_4 t} & 0 & \lambda_4 e^{-\lambda_4 t} \end{bmatrix} \tag{5.33}$$

where λ_1=0.2, λ_2=0.1, λ_3=0.4, λ_5=0.5, ϵ=2.5x10^{-6}, (all units are in sec^{-1}).

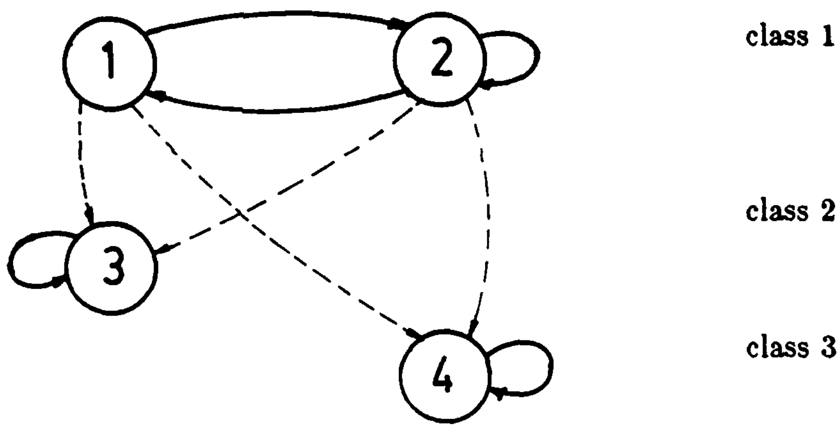


Figure 5-4: State transition diagram for Case II

In this model and those that follow, all the transition holding time density functions are exponential. However, different exponential functions are used for different destinations from each state. This renders this model, and the models that follow in this chapter, semi-Markov.

Approximate Markov process

Since the non-perturbed semi-Markov process for class 1 and the transition kernel elements for exits from class 1 are similar to those in Case I, the non-perturbed semi-Markov process stationary probability distribution for class 1 and Λ are the same. These results are repeated for convenience here,

$$\underline{\pi}^{(1)} = [\,0.2609 \quad 0.7391\,]^T ,$$

$$\Lambda = 0.9391 \,. \tag{5.34}$$

However, there will be different eventual transition probabilities from aggregated "state" 1 to aggregated "states" 2 and 3. They are evaluated as follows:

$$p_{21} = \frac{\sum\limits_{i \in E_1} \pi^{(1)}_{M_i} q^{(21)}_i}{\sum\limits_{i \in E_1} \pi^{(1)}_{M_i} q^{(1)}_i} \tag{5.35}$$

where

$$q^{(1)}_i = \sum_{j \in E_1} q^{(1)}_{ji} \tag{5.36}$$

therefore,

$$q^{(1)}_1 = 6$$
$$q^{(1)}_2 = 9$$

and,

$$q^{(21)}_i = \sum_{j \in E_2} q^{(1)}_{ji} \tag{5.37}$$

therefore,

$$q_1^{(21)} = q_{31}^{(1)} = 2$$
$$q_2^{(21)} = q_{32}^{(1)} = 6$$

substituting these quantities into Eq. (5.35), then

$$p_{21} = 0.6111 \tag{5.38}$$

Since class 1 can only transit to classes 2 and 3 :

$$p_{31} = 1 - p_{21} = 0.3889 \tag{5.39}$$

Then the transition kernel elements exiting aggregated "state" 1 are,

$$\phi_{21}(s) = 0.6111 \; \frac{0.9391}{s + 0.9391} \tag{5.40}$$

$$\phi_{31}(s) = 0.3889 \; \frac{0.9391}{s + 0.9391} \tag{5.41}$$

If the initial state probability vector is,

$$\underline{\pi}(0) = [\,1 \quad 0 \quad 0 \quad 0\,]^T \tag{5.42}$$

then the probabilities in classes 2 and 3 in scaled time t'will be approximated by:

$$\pi_2^e(t') = 0.6111\,[\,1 - e^{-0.9391\,t'}] \tag{5.43}$$

$$\pi_3^e(t') = 0.3889\,[\,1 - e^{-0.9391\,t'}] \tag{5.44}$$

or, in the original time scale,

$$\pi_2^e(t) = 0.6111\,[\,1 - e^{-0.9391 \times 2.5 \times 10^{-6}\,t}\,] \tag{5.45}$$

$$\pi_3^e(t) = 0.3889\,[\,1 - e^{-0.9391 \times 2.5 \times 10^{-6}\,t}\,] \tag{5.46}$$

Exact solution of the original semi-Markov process

The exact solutions for $\Phi_{31}(t)$ and $\Phi_{41}(t)$ are evaluated analytically with the help of MACSYMA. If the initial probability distribution is as in Eq. (5.42) then,

$$P_{E_2}(t) = \Phi_{31}(t) \tag{5.47}$$

$$P_{E_3}(t) = \Phi_{41}(t) \tag{5.48}$$

that is,

$$\begin{aligned} P_{E_2}(t) = {} & 3.15232e{-}13\, e^{-0.11500\,t} \left[-1.93859e12\, sinh(0.114998t) \right. \\ & \left. - 1.93859e12\, cosh(0.114998t) \right] - 1.02947e{-}6\, e^{-0.4t} \\ & + 0.61111 \end{aligned} \tag{5.49}$$

$$\begin{aligned} P_{E_3}(t) = {} & 3.15232e{-}13\, e^{-0.11500\,t} \left[-1.23364e12\, sinh(0.114998t) \right. \\ & - 1.23363e12\, cosh(0.114998\, t) - 8.77778e{-}6\, e^{-0.5t} \\ & + 0.38889 \end{aligned} \tag{5.50}$$

It can be seen that the interval transition probability functions from state 1 to state 3 and from state 1 to state 4 are a sum of exponential terms despite the fact that all of the holding time density functions in the model are exponential.

Comparison of results

The exact and approximate class probability results in the closed form obtained above are evaluated and compared at different time points up to 10,000 seconds in Figure 5-5. From the numerical results in the figure, it can be seen that the maximum error occurs in the fifth decimal place up to t=10,000 seconds. The approximate class probability distribution when all the probability in class 1 has moved to classes 2 and 3, can be obtained by substituting Eq. (5.45) and (5.46) with

t/sec.	exact class 2 probability $P_{E_2}(t)$	approximate class 2 probability $\pi_2^e(t)$	exact class 3 probability $P_{E_3}(t)$	approximate class 3 probability $\pi_3^e(t)$
5	0.00001	0.00001	0.00000	0.00000
50	0.00007	0.00007	0.00005	0.00005
200	0.00029	0.00029	0.00019	0.00018
1000	0.00143	0.00143	0.00092	0.00091
5000	0.00713	0.00713	0.00455	0.00454
10000	0.01418	0.01418	0.00903	0.00902

Figure 5-5: Comparison of approximate and exact classes probabilities

t=∞. The results are:

$$\pi_2^e(\infty) = 0.6111 \tag{5.51}$$

$$\pi_3^e(\infty) = 0.3889 \tag{5.52}$$

The class probability distribution at t=∞ can also be obtained from the exact solution in Eq. (5.49) and (5.50). All the terms in both equations except the constant terms will vanish when t=∞, so the class probability distribution will be,

$$P_{E_2}(\infty) = 0.6111 \tag{5.53}$$

$$P_{E_3}(\infty) = 0.3889 \tag{5.54}$$

By comparing the exact and approximate results, it can be speculated that the entire class probability history is well approximated by the approximate Markov process.

It was demonstrated in this example that the class probability trajectory is also well approximated by the approximate Markov process for a particular model where a class can transit to two different classes.

5.3 Case III

The 9-state model and the models in Cases I and II do not yield an associated aggregated model for which classes 2 or 3 can transit back to class 1. This situation would arise in models of fault-tolerant systems that include on-line repair. This provides the motivation to create a new model to demonstrate the accuracy of the approximate Markov process for this situation. The new model is similar to the one used in Case II in that class 1 can transit to both class 2 and class 3. However, class 2 can transit back to class 1 in the new model. The state transition diagram of this new model is shown in Figure 5-6. The transition kernel matrix is similar to that of Case II except that there are transitions possible back to class 1 from class 2. This yields two new nonzero off-diagonal elements in the transition kernel matrix, which is defined as follows:

$$P(t) = \begin{bmatrix} 0 & (0.3-7\epsilon)\lambda_1 e^{-\lambda_1 t} & 2\epsilon\lambda_1 e^{-\lambda_1 t} & 0 \\ (1-6\epsilon)\lambda_2 e^{-\lambda_2 t} & (0.7-2\epsilon)\lambda_2 e^{-\lambda_2 t} & 4\epsilon\lambda_2 e^{-\lambda_2 t} & 0 \\ 2\epsilon\lambda_3 e^{-\lambda_3 t} & 6\epsilon\lambda_3 e^{-\lambda_3 t} & (1-6\epsilon)\lambda_3 e^{-\lambda_3 t} & 0 \\ 4\epsilon\lambda_4 e^{-\lambda_4 t} & 3\epsilon\lambda_4 e^{-\lambda_4 t} & 0 & \lambda_4 e^{-\lambda_4 t} \end{bmatrix} \tag{5.55}$$

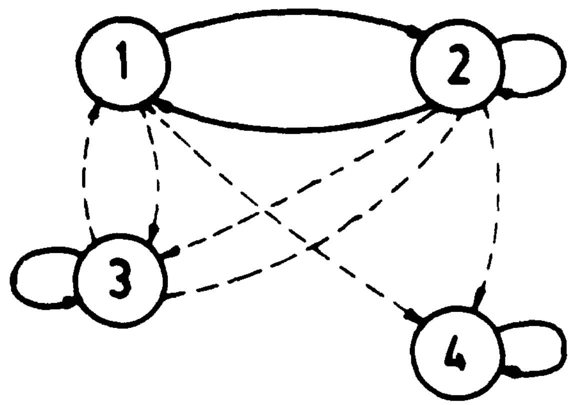


Figure 5-6: State transition diagram for Case III

where λ_1=0.2, λ_2=0.1, λ_3=0.4, λ_5=0.5, ϵ=2.5x10^{-6}, (all units are in sec^{-1}).

Stationary probability distribution of the non-perturbed semi-Markov process

The structure of the process in this case is different from that of Case II. However, the non-perturbed process in class 1 is exactly the same as that of Cases I and II. So the stationary probability distribution is the same as before, namely:

$$\underline{\pi}^{(1)} = [\,0.2609 \quad 0.7391\,]^T \tag{5.56}$$

Since classes 2 and 3 each consist of only one state, their stationary distribution are,

$$\underline{\pi}^{(2)} = [\,1\,]^T \tag{5.57}$$

$$\underline{\pi}^{(3)} = [\,1\,]^T \tag{5.58}$$

Approximate Markov process

Because of the similarity of this process with that of case II, some of the parameters of the approximate Markov process are the same, and they are stated here:

$$p_{21} = 0.6111 \qquad (5.59)$$

$$p_{31} = 0.3889 \qquad (5.60)$$

$$A_1 = 0.9391 \qquad (5.61)$$

In this model, however, class 2 can transit back to class 1, so A_2 is calculated as follows:

$$A_2 = \frac{\pi_3^{(2)} q_3^{(2)}}{\pi_3^{(2)} a_3^{(2)}} = \frac{q_3^{(2)}}{a_3^{(2)}} \qquad (5.62)$$

where

$$q_3^{(2)} = \sum_{j \in E_2} q_{j3}^{(2)} = 6$$

$$a_3^{(2)} = \sum_{j \in E_2} p_{j3}\, \tau_{j3} = p_{33}/\lambda_3 = 2.5$$

Substituting these quantities into Eq. (5.62) :

$$A_2 = 6/2.5$$
$$=2.4 \tag{5.63}$$

Class 2 transits only to class 1, therefore

$$\phi_{12}(s) = \frac{2.4}{s + 2.4} \tag{5.64}$$

Then the approximate aggregated Markov process in the new time scale is characterized by the following Laplace transformed transition kernel matrix,

$$P^s(s) = \begin{bmatrix} 0 & \frac{2.4}{(s+2.4)} & 0 \\ 0.6111\,\frac{0.9391}{(s+0.9391)} & 0 & 0 \\ 0.3889\,\frac{0.9391}{(s+0.9391)} & 0 & \frac{s}{s+1} \end{bmatrix} \tag{5.65}$$

or in the scaled time domain,

$$P^s(t) = \begin{bmatrix} 0 & 2.4e^{-2.4t} & 0 \\ 0.6111 \text{ x } 0.9391e^{-0.9391t} & 0 & 0 \\ 0.3889 \text{ x } 0.9391e^{-0.9391t} & 0 & e^{-t} \end{bmatrix} \tag{5.66}$$

If the initial condition of the process is

$$\underline{\pi}(0) = [\,1 \quad 0 \quad 0 \quad 0\,]^T \tag{5.67}$$

then,

$$\pi_i^e(t) = \phi_{i1}^e(t) \tag{5.68}$$

By using continuous time Markov theory, the interval transition probability matrix and then $\pi_1^e(t)$ in the original time scale are found to be:

$$\pi_1^e(t) = 1.0710e{-}8\; e^{-1.6696 \times 2.5\times10^{-6}\, t} \,[\, 4.9338\; sinh(1.3824 \times 2.5\times10^{-6}\, t) + 9.3373e7\; cosh(1.3824 \times 2.5\times10^{-6}\, t)\,] \tag{5.69}$$

$$\pi_2^e(t) = 4.1517e{-}1\; e^{-1.6696 \times 2.5\times10^{-6}\, t}\; sinh(1.3824 \times 2.5\times10^{-6}\, t) \tag{5.70}$$

$$\pi_3^e(t) = 1.0710e{-}8\; e^{-1.6696 \times 2.5\times10^{-6}\, t} \,[\, -8.8103e{+}7\; sinh(1.3824 \times 2.5\times10^{-6} t\,) - 9.3372e7\; cosh(1.3824 \times 2.5\times10^{-6}\, t)\,] + 0.99999 \tag{5.71}$$

Exact solution of the original semi-Markov process

The exact solution of total probability in each class in this example is calculated numerically by the same matrix convolution sum method as was used for the 9-state model. The normalized probability distribution in class 1 was calculated analytically with the help of MACSYMA. The results appear below.

Comparison of results

The results for the exact and for the approximate class probability distributions are compared in Figure 5-7. This example shows again that the exact aggregated probability distribution is *well approximated by the enlarged process* because the maximum errors occur only at the fifth decimal place.

The normalized probability distribution in class 1 is shown in Figure 5-8. It can be seen from the normalized probability distribution history in Figure 5-8 that stationarity is established after t=200 seconds When this is compared with the stationary probability distribution of the non-perturbed process in class 1 that is given in Eq. (5.56), it is found that they agree up to four decimal places.

It has been emphasized here that in this model, there are transitions possible

t/sec.	exact class 1 probability $P_{E_1}(t)$	approximate class 1 probability $\pi_1^e(t)$	exact class 2 probability $P_{E_2}(t)$	approximate class 2 probability $\pi_2^e(t)$
10	0.99997	0.99998	0.00002	0.00001
100	0.99977	0.99977	0.000014	0.00014
200	0.99953	0.99953	0.00028	0.00029
500	0.99885	0.99883	0.00070	0.00072
1000	0.99772	0.99766	0.00139	0.00143

t/sec.	exact class 3 probability $P_{E_3}(t)$	approximate class 3 probability $\pi_3^e(t)$
10	0.00002	0.00001
100	0.00010	0.00009
200	0.00018	0.00018
500	0.00045	0.00046
1000	0.00089	0.00091

Figure 5-7: Comparison of approximate and exact classes probabilities

t/sec.	state 1	state 2
0	1.00000	0.00000
10	0.4791	0.5209
50	0.26449	0.73551
100	0.26089	0.73911
200	0.26086	0.73914
500	0.26086	0.73914
1000	0.26086	0.73914

Figure 5-8: Normalized probability distribution in class 1

both out of and into class 1 and it has been shown from the results above that the semi-Markov process is well approximated in this case when the enlarged process is expanded by the stationary probability distribution of the non-perturbed process.

5.4 Case IV

For some fault-tolerant system semi-Markov models, there may be trapping states among some classes of states. Under these circumstances, the ergodicity condition in the Theorem presented in Chapter 2 will not be satisfied by these kind of models. It is of interest to know whether the approximate technique will be valid for some of these models. So, in this 4-state example, a model with two non-ergodic classes is created where each class consists of two states. The state transition diagram is shown in Figure 5-9 and the process is governed by the transition kernel matrix in Eq. (5.72).

$$P(t) = \begin{bmatrix} (0.5-\epsilon)\lambda_1 e^{-\lambda_1 t} & 0 & 0 & 0 \\ (0.5-5\epsilon)\lambda_2 e^{-\lambda_2 t} & (1-9\epsilon)\lambda_2 e^{-\lambda_2 t} & 0 & 0 \\ 2\epsilon\lambda_3 e^{-\lambda_3 t} & 6\epsilon\lambda_3 e^{-\lambda_3 t} & 0.4\lambda_3 e^{-\lambda_3 t} & 0 \\ 4\epsilon\lambda_4 e^{-\lambda_4 t} & 3\epsilon\lambda_4 e^{-\lambda_4 t} & 0.6\lambda_4 e^{-\lambda_4 t} & \lambda_4 e^{-\lambda_4 t} \end{bmatrix} \tag{5.72}$$

where λ_1=0.2, λ_2=0.1, λ_3=0.4, λ_5=0.5, ϵ=2.5x10^{-6}, (all units are in sec^{-1}).

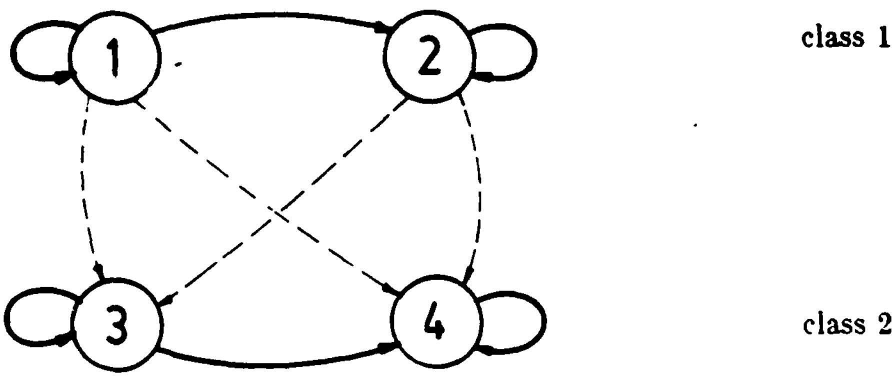


Figure 5-9: State transition diagram for Case IV

Stationary probability distribution of the non-perturbed semi-Markov process

By decomposing the transition kernel matrix, the transition probability matrix of the non-perturbed imbedded Markov process is as follows:

$$P = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0.6 & 1 \end{bmatrix} \tag{5.73}$$

By raising the transition probability matrix to successively higher powers until stationary is established, the stationary interval transition probability matrix is found to be,

$$P^{\infty} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{5.74}$$

Therefore, the stationary probability vectors of the non-perturbed imbedded Markov processes in classes 1 and 2 are:

$$\underline{\pi}_M^{(1)} = [\,0 \quad 1\,]^T \tag{5.75}$$

$$\underline{\pi}_M^{(2)} = [\,0 \quad 1\,]^T \tag{5.76}$$

Hence, the stationary probability vectors of the non-perturbed semi-Markov processes are,

$$\underline{\pi}^{(1)} = [\,0 \quad 1\,]^T \tag{5.77}$$

$$\underline{\pi}^{(2)} = [\,0 \quad 1\,]^T \tag{5.78}$$

Approximate Markov process

The Laplace transform of the transition kernel for transitions from aggregated "state" 1 to "state" 2 is given by

$$\phi_{21}(s) = p_{21} \frac{\Lambda_1}{s + \Lambda_1} \tag{5.79}$$

where

$$\Lambda_1 = \frac{\sum_{i \in E_1} \pi_{M_i}^{(1)} q_i^{(1)}}{\sum_{i \in E_1} \pi_{M_i}^{(1)} \tau_i^{(1)}} \tag{5.80}$$

From the transition kernel matrix in Eq. (5.72)

$$q_2^{(1)} = q_{22}^{(1)} = 9$$

$$\tau_2^{(1)} = p_{22}\,\bar{\tau}_{22} = 10$$

Substituting the above quantities and Eq. (5.75) into Eq. (5.80) gives

$$\Lambda_1 = \frac{0 \times q_1^{(1)} + 9}{0 \times \tau_1^{(1)} + 10} = 0.9 \tag{5.81}$$

and from the structure of the model,

$$p_{21} = 1 \tag{5.82}$$

So the transition kernel element for transition from aggregated "state" 1 to 2 in new time scale is,

$$\phi_{21}(s) = \frac{0.9}{s + 0.9} \tag{5.83}$$

or in the scaled time domain,

$$\phi_{21}(t') = 0.9\, e^{-0.9t'} \tag{5.84}$$

Because there are only two classes:

$$\pi_1^e(t') = e^{-0.9t'} \tag{5.85}$$

$$\pi_2^e(t') = 1 - e^{-0.9\, t'} \tag{5.86}$$

In the original time scale,

$$\pi_1^e(t) = e^{-0.9 \times 2.5 \times 10^{-6}\, t} \tag{5.87}$$

$$\pi_2^e(t) = 1 - e^{-0.9 \times 2.5 \times 10^{-6}\, t} \tag{5.88}$$

Exact solution of the original semi-Markov process

Exact solutions in closed form of the total probability and normalized probability distribution in class 1 were evaluated with the help of MACSYMA and they are,

$$P_{E_1}(t) = \pi_1(t) + \pi_2(t) \tag{5.89}$$

$$\bar{\pi}_1(t) = \pi_1(t)/P_{E_1}(t) \tag{5.90}$$

$$\bar{\pi}_2(t) = \pi_2(t)/P_{E_1}(t) \tag{5.91}$$

where,

$$\pi_1(t) = 1.00339\text{x}10^5\, e^{-9.99999\text{x}10^{-2}\, t} - 1.00338\text{x}10^5\, e^{-1.00000\text{x}10^{-1}\, t}$$
$$+\, 3.33334\text{x}10^{-6}\, e^{-\,4.0\text{x}10^{-1}\, t} + 7.50000\text{x}10^{-6}\, e^{-5.0\text{x}10^{-1}\, t}$$
$$-\, 5.64491\text{x}10^{-9} \qquad (5.92)$$

$$\pi_2(t) = 1.54580\text{x}10^{-12}\, e^{-5.00014\text{x}10^{-2}\, t}\, [\, 6.49386\text{x}10^{16}$$
$$\mathit{cosh}(\, 4.99991\text{x}10^{-2}\, t\,) - 6.49373\text{x}10^{16}\, \mathit{sinh}(\, 4.99991\text{x}10^{-2}\, t)\,]$$
$$-\, 1.00382\text{x}10^5\, e^{-9.99999\text{x}10^{-2}\, t} - 1.24998\text{x}10^{-6}\, e^{-\,4.0\text{x}10^{-1}\, t}$$
$$-\, 5.62498\text{x}10^{-7}\, e^{-5.0\text{x}10^{-1}\, t} - 1.47474\text{x}10^{-4} \qquad (5.93)$$

Comparison of results

The total probability of occupying class 1 obtained from the approximate aggregated Markov process and from the analytical solution of the original semi-Markov process are compared in Figure 5-10. The exact and approximate solutions listed in the figure agree to four decimal places except after one million seconds have elapsed where the error occurs in the fourth decimal place.

The exact normalized probability distribution history within class 1 is shown in Figure 5-11. It can be seen from the results in the figure that the stationary normalized probability distribution agrees with the stationary probability distribution of the non-perturbed process in class 1. The trajectory converges within less than 0.0003 absolute error after t=100 seconds.

This example, which consists of two non-ergodic classes, shows that the original process aggregated probability distribution history is well approximated by the approximate process and that the normalized probability distribution in class 1 converges to the stationary probability distribution of the non-perturbed process after a brief transient period, even though this model violates the sufficient condition stated in references [4, 5] that the non-perturbed classes be ergodic. The

t/sec.	approximate class 1 probability $\pi_1^e(t)$	exact class 1 probability $\pi_1^e(t)$
10	0.99997	0.99998
100	0.99976	0.99978
200	0.99954	0.99955
500	0.99886	0.99888
1000	0.99774	0.99775
5000	0.98880	0.98881
10000	0.97773	0.97775
1000000	0.10527	0.10540

Figure 5-10: Comparison of approximate and exact probability in class 1

t/sec.	state 1	state 2
10	0.55183	0.44817
100	0.00027	0.99973
200	0.00000	1.00000
500	0.00000	1.00000

Figure 5-11: Normalized probability distribution in class 1

implication of this example is that some non-ergodic models can be analyzed with the approximation technique. This opens a wider scope of fault-tolerant system models to be approximately analyzed by this technique.

5.5 Case V

The example model in this case, the last in this chapter, comprises four classes, and both classes 1 and 2 can transit to class 3. This situation is found in none of the models examined before. In this section, only the exact and approximate probability in class 3 will be examined. The state transition diagram is shown in Figure 5-12 and the process is characterized by the following transition kernel matrix:

$$P(t) = \begin{bmatrix} (1-60\epsilon)\lambda_1 e^{-\lambda_1 t} & 0 & 0 & 0 \\ 0 & (1-9\epsilon)\lambda_2 e^{-\lambda_2 t} & 0 & 0 \\ 20\epsilon\lambda_3 e^{-\lambda_3 t} & 6\epsilon\lambda_3 e^{-\lambda_3 t} & \lambda_3 e^{-\lambda_3 t} & 0 \\ 40\epsilon\lambda_4 e^{-\lambda_4 t} & 3\epsilon\lambda_4 e^{-\lambda_4 t} & 0 & \lambda_4 e^{-\lambda_4 t} \end{bmatrix} \tag{5.94}$$

where λ_1=0.2, λ_2=0.1, λ_3=0.4, λ_5=0.5, ϵ=2.5x10^{-6}. (all units are in sec^{-1}) States 1, 2, 3 and 4 are classes 1, 2, 3 and 4, respectively. Class 1 and 2 both transit to class 3.

Approximate aggregated Markov process

The Laplace transform of the transition kernel elements for transitions from aggregated "state" 1 to 3 and from aggregated "state" 2 to 3 are both given by:

$$\phi_{3k}(s) = p_{3k} \frac{\Lambda_k}{s + \Lambda_k} \qquad ,k = 1,2 \tag{5.95}$$

From the parameters of the transition kernel matrix in Eq. (5.94), the following

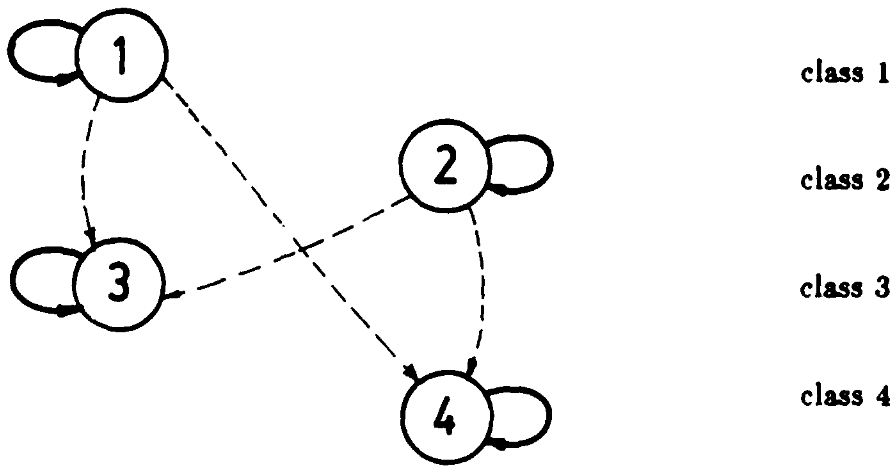


Figure 5-12: State transition diagram for Case V

approximate Markov process parameters are evaluated:

$$q_1^{(1)} = q_{11}^{(1)} = 60$$
$$q_1^{(31)} = q_{41}^{(1)} = 20$$

$$\bar{r}_1^{(1)} = p_{11}\,\bar{r}_{11} = 5$$

Therefore,

$$\Lambda_1 = \frac{q_1^{(1)}}{\bar{r}_1^{(1)}} = 12 \tag{5.96}$$

and

$$p_{31} = \frac{q_1^{(31)}}{q_1^{(1)}} = 0.3333 \tag{5.97}$$

$$q_2^{(2)} = q_{22}^{(2)} = 9$$
$$q_2^{(32)} = 6$$

$$r_2^{(2)} = p_{22}\, \bar{\tau}_{22} = 10$$

Therefore,

$$A_2 = \frac{q_2^{(2)}}{r_2^{(2)}} = 0.9 \tag{5.98}$$

and

$$p_{32} = \frac{q_2^{(32)}}{q_2^{(2)}} = 0.6667 \tag{5.99}$$

So the kernel elements for transitions to "state" 3 of the approximate aggregated Markov process are:

$$\phi_{31}(s) = 0.3333 \ \frac{12}{s + 12} \tag{5.100}$$

$$\phi_{32}(s) = 0.6667 \ \frac{0.9}{s + 0.9} \tag{5.101}$$

or, in the scaled time domain,

$$\phi_{31}(t') = 0.3333 \text{ x } 12\ e^{-12\, t'} \tag{5.102}$$

$$\phi_{32}(t') = 0.6667 \text{ x } 0.9\ e^{-0.9\, t'} \tag{5.103}$$

If the initial state probability vector is,

$$\underline{\pi}^e(0) = [\, 0.5 \quad 0.5 \quad 0 \quad 0\,]^T \tag{5.104}$$

then the probability in class 3 is,

$$\pi_3^e(t') = 0.5 \times 0.3333 \left(1 - e^{-12\, t'} \right) + 0.5 \times 0.6667 \left(1 - e^{-0.9\, t'} \right) \tag{5.105}$$

or in the original time scale,

$$\pi_3^e(t) = 10.5 \times 0.3333 \left(1 - e^{-12 \times 2.5\mathrm{x}10^{-6}\, t} \right) + 0.5 \times 0.6667 \left(1 - e^{-0.9 \times 2.5\mathrm{x}10^{-6}\, t} \right) \tag{5.106}$$

Exact solution of the original semi-Markov process

The exact solution for the probability in class 3 is evaluated analytically with the help of MACSYMA and the result is :

$$P_{E_3}(t) = 0.5 \left[-6.66655\mathrm{x}10^{-1}\, e^{-2.25\mathrm{x}10^{-6}\, t} - 1.12501\mathrm{x}10^{-5}\, e^{-4.0\mathrm{x}10^{-1}\, t} + 6.66667\mathrm{x}10^{-1} \right] + 0.5 \left[-3.33308\mathrm{x}10^{-1}\, e^{-3.0\mathrm{x}10^{-5}\, t} - 2.50019\mathrm{x}10^{-5}\, e^{-4.0\mathrm{x}10^{-1}\, t} + 3.33333\mathrm{x}10^{-1} \right. \tag{5.107}$$

Comparison of results

The approximate and exact probabilities in class 3 are compared in Figure 5-13. It can be concluded from the comparison of results in the figure that the approximate aggregated Markov process approximates well the behavior of a model in which two classes can both transit to a single trapping class.

t/sec.	exact class 3 probability $P_{E_3}(t)$	approximate class 3 probability $\pi_3^e(t)$
10	0.00008	0.00006
100	0.00059	0.00057
500	0.00287	0.00285
1000	0.00569	0.00568
5000	0.02696	0.02694
10000	0.05062	0.05061
1000000	0.46487	0.46487

Figure 5-13: Comparison of approximate and exact probability in class 3

5.6 Closure

In this chapter, five models were created and the approximte Markov process technique were further tested beyond the 9-state model. The five different cases represent a range of class to class transition structures which include transitions from ergodic class to ergodic class, transitions from one class to two different classes, two-way communicating classes, transitions from two different classes to a single class, and non-ergodic classes. All the approximate aggregated Markov processes in these five cases characterized the behavior of the exact aggregated 4-state models very well. That is, the class probability distributions were well approximated by the approximate Markov processes. Furthermore, the normalized probability distribution trajectories converged almost exactly to the stationary probability distribution of the corresponding non-perturbed semi-Markov process after a brief transient period.

In conclusion, for these five models, the state probability distributions can be well approximated by expanding the enlarged processes results with the stationary probability distributions of the non-perturbed processes. It is speculated that in general the approximate technique work well for most fault-tolerant system models. However, there are some limitations for these results to be applied to certain types of system models. These shortcomings will be discussed in the Chapter 7.

Chapter 6
Relaxation of Ergodicity Condition

The second sufficient condition stated in Chapter 2 for the approximate Markov process to be non-trivial is that the imbedded Markov chain of the non-perturbed process within each class must be ergodic. However, it was shown in Case IV in the last chapter that both elements of the approximate results can be valid even when the non-perturbed processes are both non-ergodic. Although the classes for Case IV are non-ergodic, the stationary probability distribution could be found for both classes and they are unique. This led to further investigation of the sufficient conditions for the semi-Markov processes to be approximated by the approximate technique and the result is that Korolyuk's Theorem can be modified as follows:

Theorem: If a semi-Markov process depends on a small parameter ϵ such that its state space can be partitioned according to Eq. (2.5) and is time-scaled according to Eq. (2.7) and additionally if the transition probability operators P_k for the imbedded Markov process of the k-th class of the non-perturbed semi-Markov process satisfies:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l = [\, \underline{e} \;\; \underline{e} \; . \; . \; . \; \underline{e} \,] \tag{6.1}$$

Then the aggregated semi-Markov process can be approximated by the enlarged process defined by Eq. (2.19).

Proof: The proof follows an identical line of reasoning to the proof in Chapter 2 until the point where the functions $\phi_{rk}^{(i)}(s)$ are shown to satisfy the system Eq.(2.16). The system equations can be rewritten in linear equation vector form:

$$\underline{\phi}_{rk}(s)^T = \underline{\phi}_{rk}(s)^T P_k \tag{6.2}$$

Postmultiplying the above equation by the operator P_k and using Eq. (6.2) on the result gives:

$$\underline{\phi}_{rk}(s)^T = \underline{\phi}_{rk}(s)^T P_k^2 \tag{6.3}$$

By successively postmultiplying the system of equations and replacing the left hand side by $\underline{\phi}_{rk}(s)^T$, and averaging an infinite number of these equations:

$$\underline{\phi}_{rk}(s)^T = \underline{\phi}_{rk}(s)^T \left[\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l \right] \tag{6.4}$$

Since the operator P_k defined by $p_{ji}^{(k)}$ satisfies Eq. (6.1), then, by linear equation theory, the solution of the system of equations in Eq. (6.4) is independent of the superscript, that is:

$$\phi_{rk}^{(i)}(s) = \phi_{rk}(s) \qquad ,i \in E_k \tag{6.5}$$

The remainder of the proof that the aggregated model is Markovian and the derivation of parameters of the approximate Markov process will be exactly the same as that of the remainder of the proof in Chapter 2.

This extended Theorem is a relaxation of the ergodicity sufficient condition stated earlier in Chapter 2 imposed on the semi-Markov process to be approximated.

It is of interest to find conditions under which Eq. (6.1) is satisfied. Along these lines, the following theorem is established:

Theorem :

(1) If the imbedded Markov process, which is defined by the transition operator P_k of the k-th class of the non-perturbed semi-Markov process is ergodic,

or

(2) If it is nonergodic with one and only one eigenvalue of unity,

then the operator P_k satisfies Eq. (6.1).

Proof :

(1) By the ergodic Theorem,

$$\lim_{l \to \infty} P_k^l = \Pi_k = [\, \underline{e} \;\; \underline{e} \; . \; . \; . \; \underline{e} \,] \tag{6.6}$$

and,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l = \lim_{n \to \infty} \{ \frac{1}{n} \sum_{l=1}^{r} P_k^l + \frac{1}{n-r} \sum_{l=r+1}^{n} P_k^l \} \tag{6.7}$$

where r is finite but large such that,

$$P_k^r \approx \Pi_k \tag{6.8}$$

Therefore, Eq. (6.7) can be reduced to :

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l = \lim_{n \to \infty} \frac{1}{n-r} \sum_{l=r+1}^{n} P_k^l \tag{6.9}$$

By Eq. (6.8), it follows:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l = \Pi_k = [\, \underline{e} \;\; \underline{e} \; . \; . \; . \; \underline{e} \,]$$

which proves the Theorem for this case.

(2) The operator P_k can be put in Jordan form by the following transformation:

$$P_k = T \Lambda_k T^{-1} \tag{6.10}$$

where T is a square invertible matrix with columns made up of the right eigenvectors (or generalized right eigenvectors) of the operator P_k. By a proper ordering, Λ_k has the form:

$$\Lambda_k = \left[\begin{array}{cccc|c} \lambda_1 & & & & \\ & \lambda_2 & & & 0 \\ & & \ddots & & \\ & & & \lambda_p & \\ \hline & & 0 & & J \end{array}\right]$$

where $\{\lambda_1, \ldots, \lambda_p\}$ are the unit magnitude eigenvalues and J is a Jordan form matrix containing all the eigenvalues of less than unit magnitude on its main diagonal. (This form is known to exist for a stochastic matrix P_k because the unit magnitude eigenvalues must have a full set of linearly independant eigenvectors.) Therefore:

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} P_k^l &= \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \left[T \Lambda_k T^{-1} \right]^l \\ &= T \left[\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \Lambda_k^l \right] T^{-1} \end{aligned} \tag{6.11}$$

Since Λ_k has one and only one eigenvalue of one:

$$\lim_{n \to \infty} \frac{1}{n}\sum_{l=1}^{n} \Lambda_k^l = \text{diagonal matrix with a single non-zero element of unity on its main diagonal}$$

because $\lim_{n \to \infty} \frac{1}{n}\sum_{l=1}^{n} J_k^l = 0$ and $\lim_{n \to \infty} \frac{1}{n}\sum_{l=1}^{n} \lambda_j^l = 0$ if $|\lambda_j|=1$ and $\lambda_j \neq 1$. Because P_k is a stochastic matrix, the left eigenvector appearing in the row of T^{-1} corresponding to the unit eigenvalue is $[\,\underline{1}\,]^T$. Therefore:

$$\left[\lim_{n \to \infty} \frac{1}{n}\sum_{l=1}^{n} \Lambda_k^l\right] T^{-1} = [\,\underline{0} \ldots \underline{1} \ldots \underline{0}\,]^T \tag{6.12}$$

Therefore :

$$T\left[\lim_{n \to \infty} \frac{1}{n}\sum_{l=1}^{n} \Lambda_k^l\right] T^{-1} = T[\,\underline{0} \ldots \underline{1} \ldots \underline{0}\,]^T = [\,\underline{e}\ \underline{e} \ldots \underline{e}\,] \tag{6.13}$$

That is,

$$\lim_{n \to \infty} \frac{1}{n}\sum_{l=1}^{n} P_k = T\Lambda_k^l T^{-1} = [\,\underline{e}\ \underline{e} \ldots \underline{e}\,] \tag{6.14}$$

which completes the proof.

As an illustration of the implication of the sufficient condition stated in the second Theorem, valid and invalid examples of state transition structures are in shown Figure 6-2. Note that one of the valid examples in Figure 6-2a includes periodic intraclass behavior. The invalid example in Figure 6-2b has 2 eigenvalues of one because 2 trapping sets are present in single class.

As a result, fault-tolerant system models with non-ergodic classes that satisfy

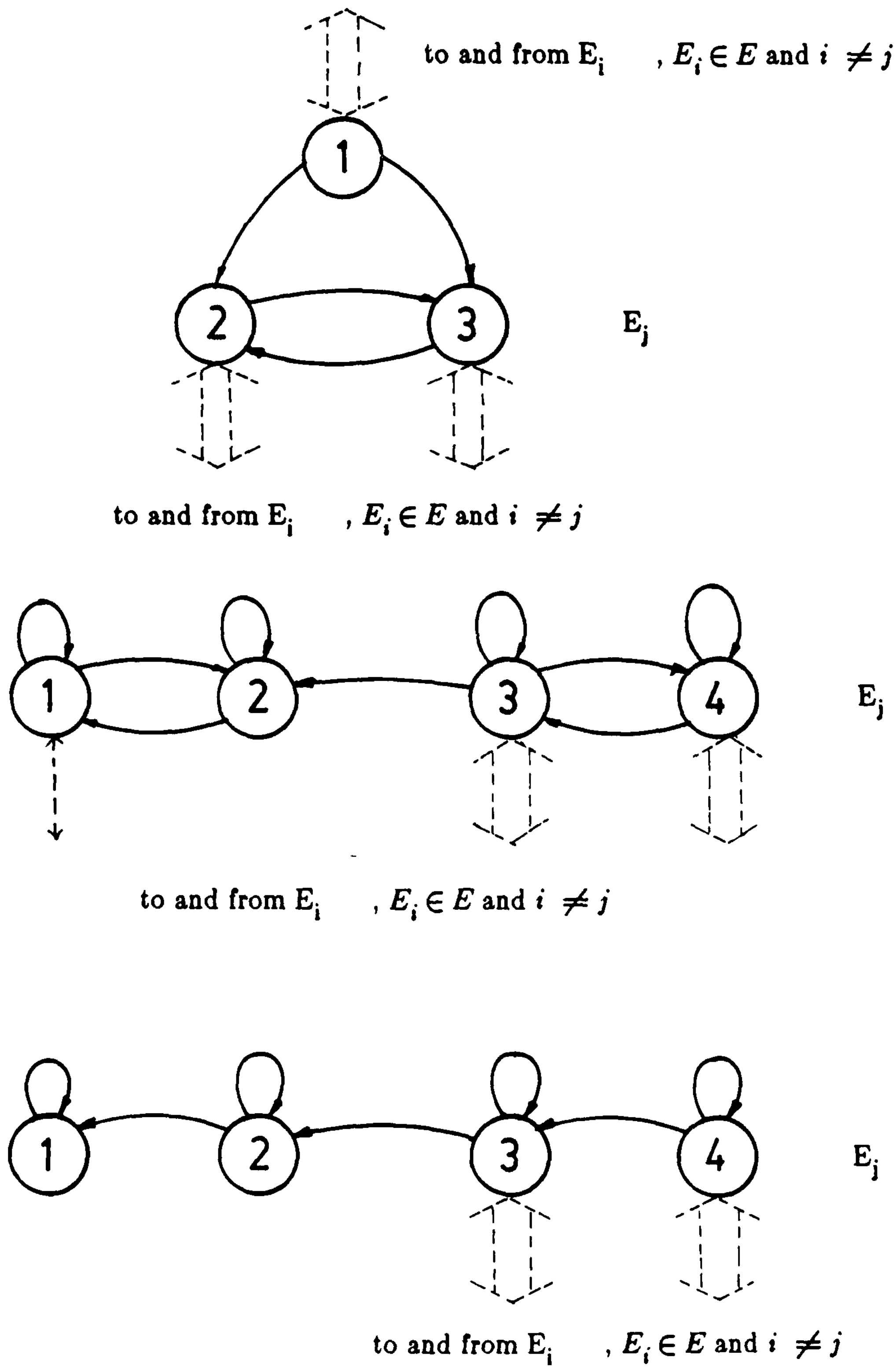


Figure 6-2a: Valid non-ergodic classes

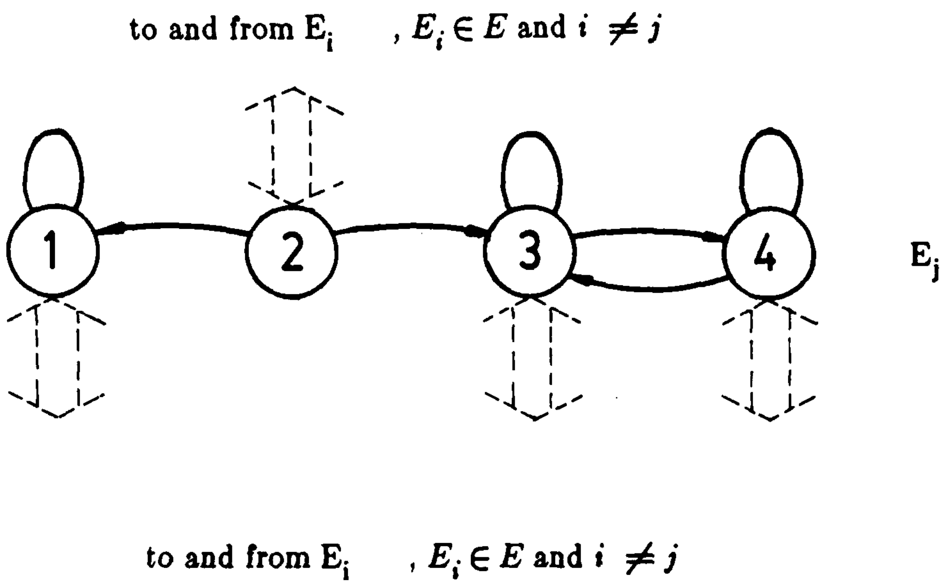


Figure: 6-2b: Invalid non-ergodic class

the condition stated in Eq. (6.1) will be approximated well by the approximation technique developed in this thesis. This explains why the approximate solution is valid for Case IV in Chapter 5. Note that there may exist fault-tolerant system models with non-ergodic classes in forms which do not satisfy Eq. (6.1) that may also be treated by the approximation technique because the Theorem is a sufficient but not necessary condition.

Chapter 7
Some Limitations on the Approximate technique

The approximate technique were demonstrated to be successful for the 9-state model and for five 4-state models in Chapters 4 and 5, respectively. The two elements that comprise the approximate technique, namely the enlarged process and the stationary probability distribution within each class, are valid for these examples. However, there are limitations for the approximate technique to be applied to certain types of system models. These limitations will be discussed in this chapter.

In the derivation of the approximate Markov process in Chapter 2, the limit of ϵ was taken to zero for the aggregated semi-Markov model to behave as a Markov process. This means that the failure rate of the instruments of a fault-tolerant system have to be small enough for the results to be well approximated by the enlarged process. One of the alternatives for investigating the values of ϵ for which the approximate Markov process diverges from the original aggregated semi-Markov model is numerical methods. The small parameter ϵ of the model in Case I in Chapter 5 was varied and the enlarged process "state" probability history was compared with the original semi-Markov model class probability history. The largest absolute error in the class 1 probability history obtained from the enlarged process is shown in Figure 7-1. It can be seen that the approximate Markov process starts to diverge when ϵ reaches the value 2.5×10^{-2}, which is one fourth of the slowest transition rate within class 1. The implication of this result is that the

ϵ	largest absolute error in class 1 probability history
2.5×10^{-6}	1.8×10^{-5}
2.5×10^{-4}	1.3×10^{-3}
2.5×10^{-3}	1.6×10^{-2}
2.5×10^{-2}	1.2×10^{-1}
5×10^{-2}	2×10^{-1}

Figure 7-1: Largest absolute error in class 1 probability history obtained from the enlarged process for the model in Case I

systems to be approximated must have a small failure rate or small perturbation parameter relative to the transition rates within each class. If the result in this particular case can be extrapolated to other cases, then ϵ cannot be larger than 1 order of magnitude smaller than the slowest intraclass transition rate.

In semi-Markov process models, the classes are often defined by the number of working and failed instruments. Occasionally, a system's system loss state is defined by different numbers of failed instruments, e.g. one wrong isolation may be as catastrophic as two uncovered failures. In these cases, the system model will contain two or more non-ergodic classes. These non-ergodic classes may not satisfy the relaxed sufficient condition defined by Eq. (6.1) and failure of the approximate technique may result.

The restrictions mentioned in this chapter limit the class of fault-tolerant system models to which the approximate technique can be applied. Note however

that for a broad class of models, the relaxed sufficient condition is satisfied and the validity of the approximate results is assured if ϵ is small enough.

Chapter 8
Summary, Conclusion and Suggestions for Further Research

8.1 Summary of Thesis

Semi-Markov models of large fault-tolerant systems whose redundancy management scheme employs sequential tests are usually intractable to practically obtain the desired length of state probability distribution histories due to the high computational cost. New methods to evaluate the state probability history of such systems in an efficient way are needed because of the growing use of complex fault-tolerant system designs.

This thesis has developed an approximate technique based on enlarged semi-Markov theory for assessing the state probability distribution histories of models of fault-tolerant systems that employ sequential tests in their fault detection and identification logic. Emphasis was placed on the extension of the theory to fault-tolerant system semi-Markov models. Secondary emphasis was placed on the demonstration of accuracy of the two elements of the approximate results, which involve expanding the enlarged process by stationary probability distributions. The use and accuracy was examined for a 9-state model and for various class to class structures that mimic fault-tolerant system models. An extended theorem, with the relaxation of the conditions that a fault-tolerant system model must satisfy for it to be approximated by the enlarged process, has been presented. Also, the limitations of the approximate technique to certain types of fault-tolerant systems was discussed.

8.2 Conclusions and Contributions

The approximate technique developed in this thesis can be used to quantify the performanceof those fault-tolerant systems with component failure rates small relative to the fault detection and isolation decision rates. This thesis has shown that the approximate technique can be a practical tool to simplify the quantification of large complex fault-tolerant system performance and might also be an efficient tool in the synthesis of such system designs.

The contributions of this thesis can be summarized as follows:

(1) Korolyuk's limit Theorem was extended by generalizing the form that the transition kernel elements may take, in which they depend through the holding time distribution on a time scale factor δ in addition to depending on the small parameter ϵ that divides the state space of the system into classes. An approximate technique based on this extended Theorem was then presented, by which the state probability history of a fault-tolerant system semi-Markov model can be approximated by expanding a reduced order Markov process state probability history by the stationary probability distributions of the non-perturbed processes within the disjoint classes. The direct benefit of this approximate technique is the reduction of the computational cost of generating results. Therefore, models of large complex fault-tolerant systems become tractable.

(2) The approximate technique has been presented here, primarily in Chapters 3 and 4, in such a way so as to illustrate its usage from the

construction of a 9-state semi-Markov fault-tolerant system model to the evaluation of the approximate solution for this model. Thus, the material in these two chapters provides an outline of the general procedures to be followed in approximating the behavior of many fault-tolerant system semi-Markov models. In addition, approximate results for five cases of different class to class transition structures for fault-tolerant system models were examined where one of these models contains two non-ergodic classes.

(3) Preliminary results were obtained for the effect of increasingly large ϵ on the error of the approximate technique.

(4) An extended theorem with the relaxation of the ergodicity condition stated in Korolyuk's original work was presented and proved in Chapter 6. As a result, the approximate technique can be applied to a wider scope of fault-tolerant system models which includes those with certain types of non-ergodic classes. Another theorem also presented in Chapter 6, establishes properties of the transition probability operator P_k of the imbedded Markov process for class k within the non-perturbed semi-Markov process which imply satisfaction of the relaxed sufficient condition.

8.3 Suggestions for Further Work

The results of the approximate technique and the limitations of it suggest possible areas to which further consideration might be given. Some of these will be listed below:

1. The realistic model that the approximate technique was applied to in

this thesis is the 9-state model described in Chapter 3. One of the assumptions in the model construction is that the failure rates of all three instruments are the same. However, in more complex systems there might be several types of instruments and each one of these may have a different small failure rate. Then the models of such systems would involve more than one perturbation parameter. The construction of an approximate technique for such systems deserves investigation.

2. There may be situations for which the semi-Markov model of a fault-tolerant system may be characterized by several different orders of mean time to transition between states. This may arise when the false alarm rate or repair rate is much slower than the fault detection and isolation decision rate or the self-test decision rate but is still much higher than the failure rate of the instruments. This gives room for the investigation of accuracy and convergence of the approximate solution for models with different combinations of relative order of perturbation parameters and two or more different orders of mean holding time distributions for transitions between states.

3. The ergodicity condition within Korolyuk's Theorem was relaxed in Chapter 6, as a result a wider class of fault-tolerant system models can be approximated by the approximate technique, but it is of interest to know how many fault-tolerant system models in real situations fall into the category of models that do not satisfy this relaxed condition. The versatility of the approximate technique can be better understood if the transition structures of general fault-tolerant system are better known.

4. In reference [5], the proof of a limit Theorem for semi-Markov processes, from which the enlarged process is deduced, depends explicitly on the

existence for each class E_k of the inverse operator $[I-P_k+\Pi_k]^{-1}$ where,

I=identity operator

P_k=transition probability operator for the imbedded Markov process of the non-perturbed semi-Markov

Π_k=cesaro limit of the multiple step transition operator associated with P_k

As is stated in [5], if E_k is an ergodic class when $\epsilon=0$ then $[I-P_k+\Pi_k]^{-1}$ is guaranteed to exist. Hence, the ergodicity of E_k is a sufficient condition for the existence of $[I-P_k+\Pi_k]^{-1}$ which in turn is a sufficient condition for the Theorem. However, ergodicity is not necessary for the existence of the inverse operator. That is, ergodicity is not necessary for the enlarged process to be valid and this was proved in the Theorem presented in Chapter 6. Further understanding of this inverse operator and the relationship with the relaxed condition may lead to further relaxation of the conditions for applying the approximate technique. This would allow application of these results to a even wider class of fault-tolerant system models.

5. The effect of nonzero ϵ on the error of the approximate results for case I of Chapter 5 was examined in Chapter 6. This provides some insight into the accuracy of the approximate solution with different ϵ for that particular example. However, this needs further investigation for other more general system models.

6. MACSYMA is a powerful symbolic manipulation tool and is also a numerical evaluation software package. Perhaps the ultimate application of the approximate technique developed here would be to

develop a MACSYMA command "program" that will input the transition kernel matrix of a fault-tolerant system model and evaluate all the non-perturbed processes stationary probability distributions and enlarged process state probability distribution histories in order to directly generate the approximate state probability histories. This package would greatly reduce the time required for reliability engineers to design or to optimize the parameters of complex fault-tolerant systems.

Appendix A
Interval Transition Probability Matrix of Semi-Markov Processes

A.1 Interval Transition Probability of Discrete Parameter Semi-Markov Processes

The following material follows that of [3].

Let a time-invariant finite state discrete parameter semi-Markov process be characterized by the transition kernel elements defined by,

$$p_{ji}(m) = p_{ji}\, h_{ji}(m) \qquad Pr\,\{\, transition\ i \rightarrow j\ occurs\ at\ sample\ m \mid state\ i\ entered\ at\ sample\ 0 \,\} \tag{A.1}$$

The first step to derive the interval transition probabilities is to consider the waiting time for each state, which is the length of time spent in a state following its entrance before a transition occurs to the same state or to a different state. In mathematical terms, if

$$w_i(m) = Pr\,\{\, waiting\ time = m \mid enter\ i\ at\ 0 \,\} \tag{A.2}$$

then,

$$w_i(m) = \sum_{j=1}^{N} p_{ji}\, h_{ji}(m) \tag{A.3}$$

In addition, if ${}^{>}w_i(n)$ denotes the waiting time in state i is greater than n samples, then

$$ {}^{>}w_i(n) = \sum_{m=n+1}^{\infty} w_i(m) \tag{A.4} $$

Now let $\phi_{ji}(n)$ is defined as the probability that the discrete-time semi-Markov process will be in state j at time n given that it entered state i at time zero. Then by considering the possible ways that the process that started by entering state i at time zero ends up in state j at time n, the following equation is reached,

$$ \phi_{ji}(n) = \delta_{ji}\, {}^{>}w_i(n) + \sum_{k=1}^{N} p_{ki} \sum_{m=0}^{n} \phi_{jk}(n-m)\, h_{ki}(m) $$

$$ i = 1,2,...N\,;\; j = 1,2,...N\,;\; n = 0,1,2,... $$

$$ \delta_{ji} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{A.5} $$

This equation can be placed in matrix form, if the following notation is adopted,

$$ W(m) = \{\, \delta_{ji}\, w_i(m) \,\}, \quad {}^{>}W(n) = \{\, \delta_{ji}\, {}^{>}w_i(n) \,\}, $$
$$ \{\, P \square H(m) \,\}_{ji} = p_{ji}\, h_{ji}(m). \tag{A.6} $$

Then by interchanging the order of summation, Eq. (A.5) can be rewritten as,

$$ \Phi(n) = {}^{>}W(n) + \sum_{m=0}^{n} \Phi(n-m)\, [\, P \square H(m) \,] \quad , \Phi(0) = I \tag{A.7} $$

A.2 Interval Transition Probability Matrix of Continuous Parameters Semi-Markov Processes

In the continuous parameter case, let the semi-Markov process be characterized by the transition kernel element defined by,

$$ p_{ji}(t) = p_{ji}\, h_{ji}(t) \tag{A.8} $$

and the waiting time and waiting time greater than t are defined by,

$$w_i(t) = \sum_{j=1}^{N} p_{ji}\, h_{ji}(t) \tag{A.9}$$

$$^{>}w_i(t) = \int_t^{\infty} w_i(\tau)\, d\tau \tag{A.10}$$

Then by similar lines of reasoning to the derivation in A.1, the continuous parameter interval transition probability can be expressed as,

$$\phi_{ji}(t) = \delta_{ji}\, {}^{>}w_i(t) + \sum_{k=1}^{N} p_{ki} \int_0^t \phi_{jk}(t-\tau)\, h_{ki}\, d\tau$$

$$i = 1,2,\ldots N \,;\; j = 1,2,\ldots N \,;\; t \geq 0$$

$$\delta_{ji} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{A.11}$$

or in matrix form,

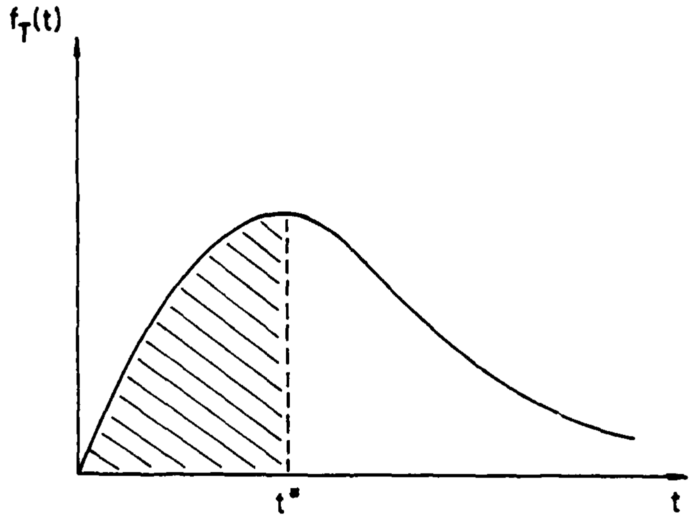
$$\Phi(t) = {}^{>}W(t) + \int_0^t d\tau\, \Phi(t-\tau)\, [\, P \square H(\tau)\,] \quad , \Phi(0) = I \tag{A.12}$$

Appendix B
Characteristic of 2nd Order Erlang Probability Density Function

An Erlang random variable T of order 2 is characterized by the following probability density function:

$$f_T(t) = \begin{cases} \lambda^2 \, t \, e^{-\lambda t} & , t \geq 0 \\ 0 & , otherwise \end{cases} \tag{B.1}$$

and a typical sketch of this function looks like the following,



The function has the following characteristics:

$\overline{T}$, expected value of the random variable T, is given by:

$$\bar{T} = E[T]$$
$$= \int_0^\infty t\, f_T(t)\, dt$$
$$= \frac{2}{\lambda} \qquad (B.2)$$

t^*, the time at which the function has a maximum value, can be evaluated by differentiating the function once and setting the result equal to zero,

$$\frac{df}{dt} = \lambda^2 e^{-\lambda t} \{ 1 - \lambda t \} = 0 \qquad (B.3)$$

Therefore,

$$t^* = \frac{1}{\lambda}$$

The cumulative probability up to time t^* is,

$$Pr\{ T < t^* \} = \int_0^{t^*} f_T(t)\, dt$$
$$= \text{shaded area}$$
$$= 0.2642 \qquad (B.3)$$

Appendix C
Transition Kernel Elements of the 9-State Model

Numerical values of parameters for the transition kernel elements:

$\lambda_0 = .001$ $\quad \lambda_{W0} = 0.05$ $\quad \lambda_{F0} = 0.1$

$\lambda_1 = 0.05$ $\quad \lambda_{W1} = 0.1$ $\quad \lambda_{F1} = 0.05$

Transition kernel elements for transitions within class 1: The p_{ji} and q_{ji} are defined by Eq. (2.2), and a_{ji} are defined by Eq. (2.26). The remaining quantities are:

$$p_{21}(t) = \lambda_0^2 \, t \, e^{-(\lambda_0 + 3\epsilon)t}$$
$$\approx \{ 1 - \frac{6\epsilon}{\lambda_0} \} (\lambda_0 + 3\epsilon)^2 \, t \, e^{-(\lambda_0 + 3\epsilon)t}$$

$p_{21} = 1$
$q_{21} = 6000$
$a_{21} = 2000$

$$p_{22}(t) = \lambda_{W0}^2 \, t \, (\lambda_{W1} t + 1) \, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t}$$
$$\approx \{ \frac{2 \lambda_{W0}^2 \lambda_{W1}}{(\lambda_{W0} + \lambda_{W1})^3} - \epsilon \frac{18 \lambda_{W0}^2 \lambda_{W1}}{(\lambda_{W0} + \lambda_{W1})^4} \}$$
$$\frac{1}{2} (\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3 \, t^2 \, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t}$$
$$+ \{ \frac{\lambda_{W0}^2}{(\lambda_{W0} + \lambda_{W1})^2} - \epsilon \frac{6 \lambda_{W0}^2}{(\lambda_{W0} + \lambda_{W1})^3} \}$$
$$(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^2 \, t \, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)t}$$

$p_{22_1} = 0.148$

$p_{22_2} = 0.111$

$p_{22} = 0.259$

$q_{22_1} = 8.888$

$q_{22_2} = 4.444$

$q_{22} = 13.333$

$a_{22_1} = 20$

$a_{22_2} = 13.333$

$$p_{32}(t) = \lambda_{W1}^2\, t\, (\lambda_{W0}\, t + 1)\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\, t}$$

$$\approx \left\{ \frac{2\,\lambda_{W1}^2\,\lambda_{W0}}{(\lambda_{W0} + \lambda_{W1})^3} - \epsilon\, \frac{18\,\lambda_{W1}^2\,\lambda_{W0}}{(\lambda_{W0} + \lambda_{W1})^4} \right\} \frac{1}{2}(\,\lambda_{W0} + \lambda_{W1} + 3\epsilon\,)^3\, t^2\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\, t}$$

$$+ \left\{ \frac{\lambda_{W1}^2}{(\lambda_{W0} + \lambda_{W1})^2} - \epsilon\, \frac{6\,\lambda_{W1}^2}{(\lambda_{W0} + \lambda_{W1})^3} \right\} (\,\lambda_{W0} + \lambda_{W1} + 3\epsilon\,)^2\, t\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\, t}$$

$p_{32_1} = 0.296$

$p_{32_2} = 0.444$

$p_{32} = 0.740$

$q_{32_1} = 17.778$

$q_{32_2} = 17.778$

$q_{32} = 35.556$

$a_{32_1} = 20$

$a_{32_2} = 13.333$

$$p_{13}(t) = p_{32}(t)$$

$$p_{23}(t) = p_{22}(t)$$

Transition kernel elements for transitions from class 1 to class 2:

$$p_{41}(t) = 3\,\epsilon\,(\lambda_0\, t + 1)\, e^{-(\lambda_0 + 3\epsilon)\,t}$$
$$= \epsilon\,\frac{3\,\lambda_0}{(\lambda_0 + 3\epsilon)^2}\,(\lambda_0 + 3\epsilon)^2\, t\, e^{-(\lambda_0 + 3\epsilon)\,t}$$
$$+ \epsilon\,\frac{3}{(\lambda_0 + 3\epsilon)}\,(\lambda_0 + 3\epsilon)\, e^{-(\lambda_0 + 3\epsilon)\,t}$$

$$q_{41_1} = 3000$$
$$q_{41_2} = 3000$$
$$q_{41} = 6000$$

$$p_{52}(t) = \epsilon\,(\lambda_{W0}\,\lambda_{W1}\, t^2 + \lambda_{W0}\, t + \lambda_{W1}\, t + 1)\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$
$$= \epsilon\,\frac{2\,\lambda_{W0}\,\lambda_{W1}}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3}\,\frac{1}{2}\,(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3\, t^2\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$
$$+ \epsilon\,\frac{(\lambda_{W0} + \lambda_{W1})}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^2}\,(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^2\, t\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$
$$+ \epsilon\,\frac{1}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)}\,(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$

$$q_{52_1} = 2.936$$
$$q_{52_2} = 6.667$$
$$q_{52_3} = 6.667$$
$$q_{52} = 16.297$$

$$p_{72}(t) = 2\,\epsilon\,(\lambda_{W0}\,\lambda_{W1}\, t^2 + \lambda_{W0}\, t + \lambda_{W1}\, t + 1)\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$
$$= \epsilon\,\frac{4\,\lambda_{W0}\,\lambda_{W1}}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3}\,\frac{1}{2}\,(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^3\, t^2\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$
$$+ \epsilon\,\frac{2\,(\lambda_{W0} + \lambda_{W1})}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^2}\,(\lambda_{W0} + \lambda_{W1} + 3\epsilon)^2\, t\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$
$$+ \epsilon\,\frac{2}{(\lambda_{W0} + \lambda_{W1} + 3\epsilon)}\,(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\, e^{-(\lambda_{W0} + \lambda_{W1} + 3\epsilon)\,t}$$

$q_{72_1} = 5.926$

$q_{72_2} = 13.333$

$q_{72_3} = 13.333$

$$q_{72} = q_{72_1} + q_{72_2} + q_{72_3}$$
$$= 32.592$$

$$p_{63}(t) = p_{52}(t)$$

$$p_{83}(t) = p_{72}(t)$$

Transition kernel elements for transitions within class 2:

$$p_{54}(t) = 0.9\, \lambda_1^2\, t\, e^{-(\lambda_1 + 2\epsilon)\, t}$$
$$\approx \{\, 0.9 - \epsilon\, \frac{0.9 \times 4}{\lambda_1} \,\}\, (\, \lambda_1 + 2\epsilon\,)^2\, t\, e^{-(\lambda_1 + 2\epsilon)\, t}$$

$p_{54} = 0.9$

$q_{54} = 72$

$a_{54} = 40$

$$p_{74}(t) = 0.1\, \lambda_1^2\, t\, e^{-(\lambda_1 + 2\epsilon)\, t}$$
$$\approx \{\, 0.1 - \epsilon\, \frac{0.1 \times 4}{\lambda_1} \,\}\, (\, \lambda_1 + 2\epsilon\,)^2\, t\, e^{-(\lambda_1 + 2\epsilon)\, t}$$

$p_{74} = 0.1$

$q_{74} = 8$

$a_{74} = 40$

$$p_{55}(t) = \lambda_{F0}^2\, t\, (\lambda_{F1}\, t + 1)\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, t}$$

$$\approx \left\{ \frac{2\,\lambda_{F0}^2\,\lambda_{F1}}{(\lambda_{F0} + \lambda_{F1})^3} - \epsilon\, \frac{12\,\lambda_{F0}^2\,\lambda_{F1}}{(\lambda_{F0} + \lambda_{F1})^4} \right\} \frac{1}{2}(\lambda_{F0} + \lambda_{F1} + 2\epsilon)^3\, t^2\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, t}$$

$$+ \left\{ \frac{\lambda_{F0}^2}{(\lambda_{F0} + \lambda_{F1})^2} - \epsilon\, \frac{4\,\lambda_{F0}^2}{(\lambda_{F0} + \lambda_{F1})^3} \right\} (\lambda_{F0} + \lambda_{F1} + 2\epsilon)^2\, t\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, t}$$

$$p_{55_1} = 0.296$$

$$p_{55_2} = 0.444$$

$$p_{55} = p_{55_1} + p_{55_2} = 0.740$$

$$q_{55_1} = 11.852$$

$$q_{55_2} = 11.852$$

$$q_{55} = q_{55_1} + q_{55_2} = 23.704$$

$$a_{55_1} = 20$$

$$a_{55_2} = 13.333$$

$$p_{65}(t) = \lambda_{F1}^2\, t\, (\lambda_{F0}\, t + 1)\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, t}$$

$$\approx \left\{ \frac{2\,\lambda_{F1}^2\,\lambda_{F0}}{(\lambda_{F0} + \lambda_{F1})^3} - \epsilon\, \frac{12\,\lambda_{F1}^2\,\lambda_{F0}}{(\lambda_{F0} + \lambda_{F1})^4} \right\} \frac{1}{2}(\lambda_{F0} + \lambda_{F1} + 2\epsilon)^3\, t^2\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, t}$$

$$+ \left\{ \frac{\lambda_{F1}^2}{(\lambda_{F0} + \lambda_{F1})^2} - \epsilon\, \frac{4\,\lambda_{F1}^2}{(\lambda_{F0} + \lambda_{F1})^3} \right\} (\lambda_{F0} + \lambda_{F1} + 2\epsilon)^2\, t\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, t}$$

$p_{65_1} = 0.148$

$p_{65_2} = 0.111$

$p_{55} = p_{65_1} + p_{65_2} = 0.259$

$q_{65_1} = 5.926$

$q_{65_2} = 2.963$

$q_{65} = q_{65_1} + q_{65_2} = 8.889$

$a_{65_1} = 20$

$a_{65_2} = 13.333$

$$p_{46}(t) = p_{65}(t)$$

$$p_{56}(t) = p_{55}(t)$$

$$p_{77}(t) = \lambda_{W0}^2\, t\,(\lambda_{W1}\, t + 1)\, e^{-(\lambda_{W0} + \lambda_{W1} + 2\epsilon)t}$$

$$\approx \left\{ \frac{2\lambda_{W0}^2 \lambda_{W1}}{(\lambda_{W0} + \lambda_{W1})^3} - \epsilon \frac{12\lambda_{W0}^2 \lambda_{W1}}{(\lambda_{W0} + \lambda_{W1})^4} \right\} \frac{1}{2}(\lambda_{W0} + \lambda_{W1} + 2\epsilon)^3\, t^2\, e^{-(\lambda_{W0} + \lambda_{W1} + 2\epsilon)t}$$

$$+ \left\{ \frac{\lambda_{W0}^2}{(\lambda_{W0} + \lambda_{W1})^2} - \epsilon \frac{4\lambda_{W0}^2}{(\lambda_{W0} + \lambda_{W1})^3} \right\} (\lambda_{W0} + \lambda_{W1} + 2\epsilon)^2\, t\, e^{-(\lambda_{W0} + \lambda_{W1} + 2\epsilon)t}$$

$p_{77_1} = 0.148$

$p_{77_2} = 0.111$

$p_{55} = p_{77_1} + p_{77_2} = 0.259$

$q_{77_1} = 5.926$

$q_{77_2} = 2.963$

$q_{77} = q_{77_1} + q_{77_2} = 8.889$

$a_{77_1} = 20$

$a_{77_2} = 13.333$

$$p_{87}(t) = \lambda_{W1}^2 \, t \, (\lambda_{W0} t + 1) \, e^{-(\lambda_{W0} + \lambda_{W1} + 2\epsilon) t}$$

$$\approx \left\{ \frac{2 \lambda_{W1}^2 \lambda_{W0}}{(\lambda_{W0} + \lambda_{W1})^3} - \epsilon \frac{12 \lambda_{W1}^2 \lambda_{W0}}{(\lambda_{W0} + \lambda_{W1})^4} \right\}$$
$$\frac{1}{2} (\lambda_{W0} + \lambda_{W1} + 2\epsilon)^3 t^2 e^{-(\lambda_{W0} + \lambda_{W1} + 2\epsilon) t}$$
$$+ \left\{ \frac{\lambda_{W1}^2}{(\lambda_{W0} + \lambda_{W1})^2} - \epsilon \frac{4 \lambda_{W1}^2}{(\lambda_{W0} + \lambda_{W1})^3} \right\}$$
$$(\lambda_{W0} + \lambda_{W1} + 2\epsilon)^2 t \, e^{-(\lambda_{W0} + \lambda_{W1} + 2\epsilon) t}$$

$p_{87_1} = 0.296$

$p_{87_2} = 0.444$

$p_{55} = p_{87_1} + p_{87_2} = 0.740$

$q_{87_1} = 11.852$

$q_{87_2} = 11.852$

$q_{87} = q_{87_1} + q_{87_2} = 23.704$

$a_{87_1} = 20$

$a_{87_2} = 13.333$

$p_{48}(t) = p_{87}(t)$

$$p_{78}(t) = p_{77}(t)$$

Transition kernel elements for transitions from class 2 to class 3:

$$P_{94}(t) = 2\,\epsilon\,(\lambda_1 t + 1)\, e^{-(\lambda_1 + 2\epsilon)t}$$
$$= \epsilon \frac{2\lambda_1}{(\lambda_1 + 2\epsilon)^2}(\lambda_1 + 2\epsilon)^2\, t\, e^{-(\lambda_1 + 2\epsilon)t}$$
$$+ \epsilon \frac{2}{(\lambda_1 + 2\epsilon)}(\lambda_1 + 2\epsilon)\, t\, e^{-(\lambda_1 + 2\epsilon)t}$$

$$q_{94_1} = 2000$$

$$q_{94_2} = 2000$$

$$q_{94} = q_{94_1} + q_{94_2} = 4000$$

$$P_{95}(t) = 2\,\epsilon\,(\lambda_{F0}\lambda_{F1} t^2 + \lambda_{F0} t + \lambda_{F1} t + 1)\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)t}$$
$$= \epsilon \frac{4\lambda_{F0}\lambda_{F1}}{(\lambda_{F0} + \lambda_{F1} + 2\epsilon)^3}\frac{1}{2}(\lambda_{F0} + \lambda_{F1} + 2\epsilon)^3\, t^2\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)t}$$
$$+ \epsilon \frac{2(\lambda_{F0} + \lambda_{F1})}{(\lambda_{F0} + \lambda_{F1} + 2\epsilon)^2}(\lambda_{F0} + \lambda_{F1} + 2\epsilon)^2\, t\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)t}$$
$$+ \epsilon \frac{2}{(\lambda_{F0} + \lambda_{F1} + 2\epsilon)}(\lambda_{F0} + \lambda_{F1} + 2\epsilon)\, e^{-(\lambda_{F0} + \lambda_{F1} + 2\epsilon)t}$$

$$q_{95_1} = 5.926$$

$$q_{95_2} = 13.333$$

$$q_{95_3} = 13.333$$

$$q_{95} = q_{95_1} + q_{95_2} + q_{95_3} = 32.593$$

$$p_{96}(t) = p_{95}(t)$$

$$P_{97}(t) = 2\,\epsilon\,(\lambda_{W0}\lambda_{W1}t^2 + \lambda_{W0}t + \lambda_{W1}t + 1)\,e^{-(\lambda_{W0}+\lambda_{W1}+2\epsilon)t}$$

$$= \epsilon\,\frac{4\,\lambda_{W0}\lambda_{W1}}{(\lambda_{W0}+\lambda_{W1}+2\epsilon)^3}\,\frac{1}{2}\,(\lambda_{W0}+\lambda_{W1}+2\epsilon)^3\,t^2\,e^{-(\lambda_{W0}+\lambda_{W1}+2\epsilon)t}$$

$$+\ \epsilon\,\frac{2(\lambda_{W0}+\lambda_{W1})}{(\lambda_{W0}+\lambda_{W1}+2\epsilon)^2}\,(\lambda_{W0}+\lambda_{W1}+2\epsilon)^2\,t\,e^{-(\lambda_{W0}+\lambda_{W1}+2\epsilon)t}$$

$$+\ \epsilon\,\frac{2}{(\lambda_{W0}+\lambda_{W1}+2\epsilon)}\,(\lambda_{W0}+\lambda_{W1}+2\epsilon)\,e^{-(\lambda_{W0}+\lambda_{W1}+2\epsilon)t}$$

$$q_{97_1} = 5.926$$

$$q_{97_2} = 13.333$$

$$q_{97_3} = 13.333$$

$$q_{97} = q_{97_1} + q_{97_2} + q_{97_3} = 32.593$$

$$p_{98}(t) = p_{97}(t)$$

Since class 3 is a trapping class and consists of only one state, the form of the tarnsition kernel is not important.

Appendix D
Stationary Probability Distribution of the Non-perturbed Semi-Markov Chain in Class 2

By using the Eq. (4.10), the mean holding time for each transition for the non-perturbed process in class 2 are :

$$\bar{\tau}_{54} = \frac{2}{\lambda_1} = 40$$

$$\bar{\tau}_{74} = \bar{\tau}_{54} = 40$$

$$\bar{\tau}_{55} = \frac{p_{55_1}}{p_{55}} a_{55_1} + \frac{p_{55_2}}{p_{55}} a_{55_2} = 16$$

$$\bar{\tau}_{65} = \frac{p_{65_1}}{p_{65}} a_{65_1} + \frac{p_{65_2}}{p_{65}} a_{65_2} = 17.143$$

$$\bar{\tau}_{46} = \bar{\tau}_{65} = 17.143$$

$$\bar{\tau}_{56} = \bar{\tau}_{55} = 16$$

$$\bar{\tau}_{77} = \bar{\tau}_{22} = 17.143$$

$$\bar{\tau}_{87} = \bar{\tau}_{32} = 16$$

$$\bar{\tau}_{48} = \bar{\tau}_{87} = 16$$

$$\bar{\tau}_{78} = \bar{\tau}_{77} = 17.143$$

So the meaning holding time in each state unconditioned on the destination is calculated by Eq. (4.9) and is given as follows :

$$\bar{\tau}_4 = 40$$

$$\bar{\tau}_5 = 16.296$$

$$\bar{\tau}_6 = 16.296$$

$$\bar{\tau}_7 = 16.296$$

$$\bar{\tau}_8 = 16.296$$

Then, the mean holding time of the non-perturbed process, as defined by Eq. (4.8), in class 2 is

$$\bar{\tau} = 17.601$$

By using Eq. (4.7), the stationary probability distribution in class 2 of the non-perturbed process is found to be,

$$\pi_4 = 0.12501$$
$$\pi_5 = 0.68199$$
$$\pi_6 = 0.17684$$
$$\pi_7 = 0.00929$$
$$\pi_8 = 0.00689$$

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